MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

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Dr. A. C. D. CROMMELIN, PRESIDENT, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Miss Allie Vibert Douglas, M.B.E., Ph.D., Lecturer in Astrophysics, McGill University, Montreal, P. Quebec, Canada ;

James J. Marshall, LL.D., Secretary, Southern Cross Astronomical Society, Bank of Bay Biscayne Bldg., Miami, Florida, U.S.A. ;

Jan Hendrik Oort, Sterrewacht, Leiden ;

John R. Owens, Ffospilcorn, Llanrhystyd, Aberystwyth, Wales ; and

Thomas Routledge, B.Sc., School House, Lees Hill, Low Row, Carlisle.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Ernest Reginald Adams, B.Sc., 124 Croxted Road, West Dulwich, London, S.E. 21 (proposed by F. Addey) ;

T. Arnold Brown, M.A., B.Sc., Professor of Mathematics, University College of the South West of England, Exeter (proposed by W. J. S. Lockyer) ;

Donald Cameron-Swan, F.R.P.S., F.S.A. (Scot.), Captain, late R.A.F., Strathmore, Kalk Bay, C.P., South Africa (proposed by H. Spencer Jones) ;

Alfred George Course, Captain, Merchant Navy, Assistant Dock-master to Port of London Authority, 5 The Gardens, Tilbury, Essex (proposed by F. D. Addyman) ;

Thomas George Cowling, 52 Lyndhurst Drive, Leyton, London, E. 10 (proposed by S. Chapman) ;

George A. Davis, Jr., 936 Ellicott Square, Buffalo, New York, U.S.A. (proposed by C. A. R. Lundin) ;

Hugh Dawson, The Southern Cross Assurance Co., Ltd., Wellington, New Zealand (proposed by C. E. Adams) ;

Vincenzo Consolato Antonino Ferraro, 23 Princes Park Avenue, Golders Green, London, N.W. 11 (proposed by S. Chapman) ;

Percy Joseph Harris, B.Sc., Hillcroft, Shrewsbury Avenue, Market Harborough, Leicestershire (proposed by R. Ballard) ;

Eric Ludlow Hawke, M.A., F.R.Met.Soc., Caenwood, The Valley Road, Rickmansworth, Herts (proposed by C. O. Bartrum) ;

Christian Victor Charles Herbert, B.A., 53 Sloane Gardens, London, S.W. 1 (proposed by C. C. L. Gregory) ;

William Moffat Inverarity, M.A., B.Sc., 11 Ravenshall Road, Glasgow, S. 1 (proposed by T. L. MacDonald) ;

Derek Ainslie Jackson, Clarendon Laboratory, University Museum, Oxford (proposed by C. V. Jackson) ;

Floris Jansen, Instituto Geográfico Militar, Gral. Paz 1314, Buenos Aires, República Argentina (proposed by C. D. Perrine) ;

L. W. P. Lewis, Solva, Mawgan Porth, St. Columb, Cornwall (proposed by F. W. Dyson) ;

Cyrus Augustus Macfarlane, Ardath, Venner Road, Annerley, Brisbane, Australia (proposed by J. Beebe) ;

Donald Glenoe McIntyre, Civil Service Club, Cape Town, S. Africa (proposed by H. Spencer Jones) ;

Rev. William Isaac H. Marr, M.A., B.Sc., St. James's Manse, Kilmacolm, Renfrewshire (proposed by J. J. Ross) ;

V. V. Narliker, B.A., Fitzwilliam House, Cambridge (proposed by W. M. Smart) ;

Frank Potter, 8 Ranelagh Gardens, Hurlingham, London, S.W. 6 (proposed by J. Silas Evans) ;

Donald Harry Sadler, B.A., 6 Wilmore Road, Forest Hill, London, S.E. 23 (proposed by L. J. Comrie) ;

Russell Sullivan, B.A., 1431 North Meridian Street, Indianapolis, Indiana, U.S.A. (proposed by B. R. Baumgardt) ;

Noel Thomas, Heather Mount, Borough Green, Kent (proposed by T. Mary Lockyer) ;

Arthur Wilfred Tucker, M.A., B.Sc., Training College, Dunedin, New Zealand (proposed by J. E. Begg) ;

Rev. Walter Walters, The Rectory, Scottsdale, Tasmania, Australia (proposed by J. Brittain) ; and

Francis John Welsh Whipple, Sc.D., F.Inst.P., Director of the Kew Observatory, 6 Addison Road, Chiswick, London, W. 4 (proposed by W. M. H. Greaves).

Two hundred and fourteen presents were announced as having been received since the last meeting, including :—

Astrographic Catalogue :—Catania R. Observatory, vol. 1, pts. 3 and 4 ; vol. 2, pts. 3 and 4 ; vol. 3, pts. 3 and 4 ; Hyderabad, Nizamiah Observatory, vol. 7 ; Sydney Observatory, vols. 11, 12.

Astrographic Charts :—San Fernando Observatory, 10 charts (zone $-3°$) ; Uccle, R. Observatory, 30 charts (zones $+33°$, $+35°$, $+37°$, $+39°$.

Barlow's Tables, 3rd edition, edited by L. J. Comrie ;

Barlow and Bryan, Elementary Mathematical Astronomy, 4th edition, edited by A. C. D. Crommelin ;

Cambridge, Solar Physics Observatory, Annals, vol. 2, pt. 1 ;

Chicago, Adler Planetarium, 11 photographs of (presented by Prof. P. Fox) ;

Jeans (Sir James), The Mysterious Universe ;

New York, American Geographical Society, School of Surveying, Star Lists . . . for each degree of Latitude from 60° north to 60° south.

Warner and Swasey Company, Bronze Medal in commemoration of the 50th anniversary of the Company, 1880–1930.

The President referred to the death of Dr. E. B. Knobel and Professor H. H. Turner, and read the following resolutions of Council :—

" The Council desire to express their grief at the death of Dr. Edward Ball Knobel, who had been a Fellow of the Society for fifty-seven years. He had served on the Council from 1876 to his death, with the exception of the year 1922–1923. He had filled in succession every office in the Society, having been President in 1892 and again in 1900, Treasurer for fourteen years in the aggregate, Secretary from 1882 to 1892, and frequently Vice-President. His extensive knowledge of astronomical literature was placed at the service of the Library Committee. His labour in editing the ancient catalogues of Ptolemy and Ulugh Beg, and in making oriental astronomical works accessible to western readers deserves the gratitude of astronomers.

" The Council request the President to convey their sympathy to his sons and daughters in their bereavement, and to assure them of their sense of the great value of the services that Dr. Knobel rendered to the Society."

" The Council desire to express their grief at the death of Professor Herbert Hall Turner, who had been a Fellow of the Society for forty-five years. He served continuously on the Council from 1887 till his death, having been Secretary from 1892 to 1898, President from 1903 to 1905, Foreign Secretary from 1919 to 1930, and frequently Vice-President.

" His work in connection with the Astrographic Catalogue, both in completing the Oxford Zone and in co-ordinating and encouraging the

work of other observatories, was perhaps the most important of his many activities.

"The Council request the President to convey to his widow and daughter their sincere sympathy in their bereavement, and their assurance of the high esteem in which Professor Turner was held by the Society."

The President also referred to the death of Father Hagen, S.J., Associate of the Society.

The Analysis of Stellar Structure.

By E. A. Milne, F.R.S.

1. The present paper brings forward considerations which compel a drastic revision in our views as to the structure of the stars. According to the current theory * the ordinary stars (giants and dwarfs) are masses of perfect gas with central temperatures of the order of 10^7 degrees and with central densities of the order of 50 times the mean density; and the current theory fails entirely to account for "white dwarfs." The present paper shows that a perfect-gas star in a steady state is in nature an impossibility, and that actual stars must either possess a small but massive core of exceedingly high density and temperature, or else must be almost wholly (that is, save for a gaseous fringe) at a very high density. According to their luminosity they must be either "centrally-condensed" or "collapsed." This appears to be a fundamental property of steady-state configurations, and it corresponds to the observed division of the stars into "ordinary stars" (giants and dwarfs) and "white dwarfs." This result is not a consequence of any special hypothesis, but flows naturally from the method of analysis. The division of "ordinary stars" into "giants" and "dwarfs" would appear to be less fundamental and not to indicate any special difference of structure.

The present paper is concerned entirely with the analysis of stellar structure, but it will be followed by another on the probable origin of stellar energy and the mechanism of its evolution. Provisionally, however, we may locate the source of stellar energy in the central condensation in the case of ordinary stars; for white dwarfs, the source may be assumed to be fairly uniformly distributed through the star, owing to the approximate homogeneity. The temperature to be assigned to the central condensation appears to be of the order of 10^{10} degrees or higher. This is sufficiently high † to be consistent

* By the current theory I mean that of which the researches of Sir A. S. Eddington are the basis. It would give this paper a controversial air wholly foreign to its purpose if I were to mention Eddington's name in every place where the underlying ideas of the present investigation differ from his or where the results differ from his.

† *Cf.* Jeans, *Nature*, 1928 November 3, Supplement.

with Jeans's hypothesis of the generation of energy by the mutual annihilation of protons and electrons.* As to the mechanism of energy generation, we may form a general idea of the process as follows. If the mutual destruction of a proton and an electron is a reversible process (as it is on Dirac's theory of the proton) there will be in the central condensation a region of approximately thermodynamic equilibrium. In true thermodynamic equilibrium the number of annihilations per second (with evolution of radiation) would equal the number of syntheses of protons and electrons (with absorption of radiation), and there would accordingly be a certain definite concentration of matter (grams per cm.^{3}) at each temperature. If the temperature were decreased, the concentration of matter would decrease to preserve equilibrium, for the annihilation process is an " exothermic " one in the sense of chemistry.† Now the natural process of cooling of the stars to space will result in a slight tendency of the central condensation to cool, and this cooling, being accompanied by a decrease in the concentration of matter, will also be accompanied by an output of radiation. There is thus no explosive tendency for matter to be annihilated ; it is simply that the natural tendency to cool unlocks the energy, and the whole process of stellar radiation is analogous to the process of the cooling of a body of very high specific heat. I do not propose to examine the process further in this paper, but the above is sufficient to indicate the lines on which development may proceed. I wish to make it clear that the question of the physics of the energy-generating process and the question of stellar equilibrium are two separate matters in the first instance, and we can only make progress with either by so considering them *in the first instance.* We first make an analysis of stellar equilibrium ; we then use our analysis to elucidate the physics of energy-generation ; lastly, we must make a grand synthesis and consider the two together. The present paper is almost entirely devoted to the analysis of equilibrium.

The difficulty of a discrepancy between a supposed " astronomical " value of stellar opacity and the value predicted by pure physics is not encountered in the present analysis, at least in the form in which it is encountered in the current theory. In the current theory the opacity at the centre of a star is computed from mass and luminosity only, but the present analysis recognises at the outset that there can be no relation in nature (depending on steady-state considerations only and apart from the intrinsic physics of energy-generation) connecting simply mass, luminosity, and opacity. The three can have arbitrary values. In the present analysis, the radius of a steady-state configuration *for one given source-distribution* is determined in terms of mass, luminosity, and opacity, so that presumably comparison with observed masses, luminosities, and *radii* will ultimately evaluate the opacity. But the radius (or density, or effective temperature) now plays a fundamental

* It is also sufficiently high for the synthesis of radioactive elements. *Cf.* Rutherford, *Nature*, 1929 March 2, p. 313.

† *Cf.* Stern, *Trans. Faraday Soc.*, **21**, 477, 1925 ; Tolman, *Proc. Nat. Acad. Sci.*, **12**, 670, 1926 ; Zwicky, *ibid.*, **14**, 592, 1928 ; Kassel, *Phys. Rev.*, **35**, 778, 1930.

part instead of appearing, as in certain parts of the current theory, as a mere correction. Further, the present analysis shows that the density-distribution depends on the opacity at all points and not merely on the opacity in the far interior. Lastly, the computed opacity will depend on the source-distribution, and this can only be ascertained by discussion of the physics of energy-generation.

Thus the present analysis removes two outstanding difficulties encountered by the current theory, namely, those associated (*a*) with the existence of white dwarfs, (*b*) with the adequacy of stellar conditions for the generation of energy and the location of the source of the energy. And it radically alters the opacity question. Further, it makes a suggestion as to the origin of novæ. As for the observed mass-luminosity correlation, it shows that this cannot be established by equilibrium considerations only, but must depend upon the physics —the intrinsic nature—of the energy-generating process.

2. The considerations here presented, as regards the method of analysis of stellar structure, are a natural development of ideas originally expressed in two previous papers in *Monthly Notices* * and in two letters to *Nature*.† The results of the newer investigations were announced in a preliminary form in two letters to periodicals,‡ but the present paper goes very much further. The conceptions underlying the very formulation of the problems to be solved differ fundamentally from those employed by other investigators, and, owing I suppose to their unfamiliarity, they appear to have been frequently misunderstood.§ Owing to these misunderstandings it seems desirable to restate the ideas *de novo*, and the present paper is therefore written so as to be formally independent of its predecessors.

3. *Fundamental Ideas.*—When we observe a star, we see a mass of material radiating energy to space. Its state appears to be constant in time.‖ We therefore form to ourselves the conception of a mass of material provided in its interior with sources of energy (a sort of heating apparatus), and radiating to space from its outer layers at a rate equal to the supply of energy from the internal sources. When each particle of the mass of material remains constant in position ¶ and constant in

* *M.N.*, **90**, 17, 1929; 678, 1930.

† *Nature*, **125**, 453, 1930 March 22; 708, 1930 May 10.

‡ *Nature*, **126**, 238, 1930 August 16; *Observatory*, **53**, 238, 1930.

§ Eddington, *M.N.*, **90**, 268 and 279, 1930; *Nature*, **125**, 489, 1930 March 29; *M.N.*, **90**, 808, 1930; *Observatory*, **53**, 208, 1930; Vogt, *Veröff. d. Univ. Sternw. zu Jena*, Nr. 3, 1930. Larmor's series of communications (*Nature*, **125**, 273, 1930 February 22; *Observatory*, **53**, 113, 167, and 249, 1930), whilst nowhere misrepresenting my work and often elucidating the argument, does not, I think, recognise the fundamental point, namely, the independence of luminosity and mass as regards steady-state considerations and so the altered formulation of the problem. All these writers are preoccupied with the mass-luminosity law, which the present paper (as well as my former ones) shows to be capable of discussion only after the structure of the star has been ascertained and the physics of energy-generation established. The whole of Eddington's criticisms may be dismissed as beside the point. See *Observatory*, **53**, 305, 1930.

‖ Variable stars are excluded from this discussion.

¶ Rotating stars are not considered in this paper.

temperature, the mass is said to be in a "steady state." By whatever mechanism each element of the material transfers energy to its surroundings, its net loss of energy is made up each second by its internal heating apparatus. In the outer parts of an actual star the source strength may be taken to be zero; the net loss (emission less absorption) is then zero, and the state is one of strict radiative equilibrium.

This picture is of course an idealisation. If the source of energy is the conversion of mass into radiation, the mass is decreasing, and gravity at any fixed point is changing, so that the material will be slowly expanding or contracting. This, however, is a secondary effect, requiring stellar hydrodynamics for its discussion. It is a necessary process to relegate such secondary effects to later occasions and discuss first our ideal picture of a mass in a steady state. We must lay the foundations of the subject by making a sound study of stellar equilibrium.

For any such aggregate of material in a steady state we can observe these quantities: its mass M, its luminosity or total source-strength L, and its radius r_1. The first problem is, *What is the structure of this aggregate when in a steady state?* In other words, how can we arrange, inside a prescribed spherical volume, a prescribed mass M in such a way that each particle remains constant in position and temperature and that the total radiation to space is the prescribed amount L per second? A second problem is, Are there any necessary relations between L, M, and r_1 implied merely by the conditions of equilibrium?

The current theory does not formulate the problem in this way. It endeavours to *calculate* L from steady-state considerations only, *i.e.* from general physics without any knowledge of the nature of the energy-generating process. It does this by first making a hypothesis about the state of aggregation of the whole mass (perfect-gas hypothesis), and then claiming to set up a relation between mass, luminosity, and opacity for any given relative distribution of source-strength.*

But it is immediately obvious that no such relation can exist in nature in virtue of steady-state considerations only, *i.e.* without the inclusion of the physics of the energy-generating process. Mass, luminosity, and opacity must be in nature three independent variables capable of having arbitrarily prescribed values, *as far as steady-state considerations go.* This can be seen as follows. Imagine an arbitrary mass M, of given arbitrary opacity, endowed with sources totalling to an arbitrary strength L, left to itself in space. If L is too large, the mass will burst under its own radiation-pressure. If L does not exceed some definite limit, it will, however, simply heat up or cool down until it arrives at a steady state, when its surface emission will equal its prescribed source-strength L. It must arrive sooner or later at a steady state—there is nothing else left for it to do. L may be varied arbitrarily (provided it does not get too large) and the result still holds. Thus steady states exist for arbitrary values of L, M, and the opacity. There is thus no necessary relation between them, as far as steady-state considerations go. In the course of the adjustment of its surface

* I refer, of course, to Eddington's formula $L=4\pi cGM(1-\beta)/a\kappa_e$ (*Internal Constitution of the Stars*, p. 124, formula (90·1)).

radiation to the prescribed source-strength L, the material may have to change its state of aggregation ; thus we cannot prescribe this beforehand. If a given hypothesis as to the state of aggregation leads to a definite relation between L, M, and the opacity, then this hypothesis must be rejected as not necessarily representing Nature, but only representing her when L, M, and the opacity happen to have values which satisfy the relation found.

Thus the current theory, in claiming to establish a mass-luminosity-opacity relation, violates an inference we have made simply by the exercise of physical insight. This inference shows that one of the principal difficulties of the current theory is due to an error in the theory. The current theory predicts luminosities some 10 times bigger than are observed, or alternatively attributes the discrepancy to an error in the value of the opacity predicted by pure physics. But the simple physical considerations just advanced show that it is impossible to derive L from M and opacity, or opacity from L and M, simply from steady-state conditions ; for no relation between them can exist in nature in virtue of steady-state conditions only. From one point of view the current theory makes this mistake because it introduces as a hypothesis what should be the object of the investigation. From another point of view it makes the mistake, as we shall see later, because it picks out one special solution of the equations instead of the general one ; or again, because it fails to pick out that solution implied by the given L, M, and opacity. The selection of the appropriate solution is made by using a boundary condition which, though used in the current theory, secures a different interpretation therein.

We fully recognise that the stars in nature do not have arbitrary luminosities for given mass. We infer that their luminosities depend on something beyond steady-state considerations. Having made our analysis of the steady-state configurations for arbitrary L and M, we proceed to apply it to the particular L's and M's occurring in nature. We thus use our analysis to disclose something as to the physical state of the different types of stars. Having arrived at a picture of the stellar interior as it must be in order to have mass M and emit L, we can then attempt to locate the origin of the energy and to discuss the mode of its liberation. If we can determine its actual output, under the conditions disclosed, by means of the physics of the energy-generating process, we may be able to link up L with M and establish on theoretical grounds the observed mass-luminosity correlation. A mass-luminosity-opacity relation may even appear, but it will be something fundamentally different from that of the current theory, for it will depend on subatomic physics. We thus see that the mass-luminosity law of nature is something much more deep-seated than the observed forms of stellar structures ; the latter are the consequence of equilibrium requirements, the former depends on the physics of energy-generation.

It will be seen that, in Bohr's language, we first perform an act of renunciation—we form an analysis of steady states in which L and M are *independent* variables. We then pick out those steady states which correspond to the L's and M's occurring in nature ; we are rewarded

with insight into the true structure of the stars. This knowledge enables us to locate the energy sources and so make progress towards the problem of the *dependence* of L on M. It is essential to recognise the difference between the formal independence of L and M as regards steady-state considerations and the observed correlation of L with M in actual stars.

Thus the fundamental problem is, What is the complete series of steady-state configurations of an arbitrary mass M, evolving an arbitrary amount of energy L per second, as L varies from zero upwards, M remaining constant ?

The present paper gives a substantial though not complete answer to this problem. The identification of the configurations disclosed by the analysis with the observed structures in nature is carried out not so much from the numerically observed values of L and M as from considerations of stability.

It is remarkable that the method pursued with such conspicuous success in the combined researches of Maclaurin, Jacobi, Darwin, Liapounoff, Poincaré, and Jeans in the study of the steady-state configurations of a rotating liquid (namely, their arrangement in linear series for varying angular velocity, ω) has not previously been attempted for the star at rest. Here L plays the part of ω—it is a variable of unrestricted range.

4. *Method of Attack.*—From what is known of physics we may be reasonably sure that stellar energy (in a star in a steady state) is not being generated in the outer layers of the star. Hence the structure of the outer layers is that of a " point-source " model, for the flux $L/4\pi r^2$ and the value of gravity $GM(r)/r^2$ at r are independent of the relative *distribution* of sources or of mass inside r. As we penetrate inwards we shall sooner or later encounter the region where the energy is being generated. It is therefore suggested to us that we should examine for definiteness source-distributions consisting of an energy-generating region surrounded by a " point-source zone." Inside the energy-generating region the source-distribution is unknown, and it is one of the ultimate aims to determine it. This, however, is a problem for the physics of energy-generation. In dealing with the fundamental problem of the steady-state for given L and M, we must examine all conceivable, or at least all plausible, source-distributions. The most representative one on which to begin operations is that of the energy-generating core with *uniform* source-strength surrounded by a " point-source " zone free from sources.* There are two limiting cases of this type of distribution, namely (1) that in which the core of uniform source-strength extends throughout the configuration; (2) that in which the point-source zone extends throughout the configuration. All other source-distributions are in a sense intermediate between these two, and we " bracket " them by considering these two in detail. Let ϵ be the source-strength. Then in (1) ϵ = constant throughout the configuration ; in (2) ϵ = 0 save at the centre, where there is a point-source of strength L.

* *I.e.*, the outer layers of a point-source model.

Let κ be the coefficient of absorption. Since opacity, mass, and luminosity are independent variables so far as steady-state investigations are concerned, we give priority of place to the examination of configurations in which the opacity law is κ = constant; modifications in the opacity law can be considered subsequently. The law κ = constant will give us insight into the general nature of steady-state configurations. We shall call configurations for which both ϵ = constant and κ = constant, configurations built on the "standard model." * The standard model and the point-source model will be found to have series of steady-state configurations so similar that we may without difficulty infer the general nature of all the configurations for models of intermediate type.

Actual stars are observed to have gaseous outer layers. We therefore analyse the structure of our steady-state configurations by working inwards shell by shell from a gaseous exterior. We are entitled to assume the gas-laws to go on holding until we reach conditions, if ever, which show that they have broken down. We simply have to integrate the differential equations of the model from the boundary inwards, making use of the equation of state of a perfect gas. The integration of the differential equation requires certain boundary conditions. One of these is provided by the fact that at the boundary the mass enclosed is M and the total emission is L. Another is the familiar one that at the boundary the density vanishes and the temperature vanishes.† These boundary conditions are sufficient to select that one which we require out of the family of solutions of the differential equations.

We now come to the crucial argument of the present paper. Once the differential equations have been solved, we can as an analytical problem determine the function $M(r)$, the mass enclosed inside any radius r. We have already arranged that $M(r) \to M$ as $r \to r_1$, the boundary radius. As an analytical solution of the differential equations the function $M(r)$ exists from $r = r_1$ to $r = 0$. We shall show that the physical nature of the configurations can be deduced from the analytical behaviour of the function $M(r)$. It will be found that this behaviour discloses a certain characteristic feature of the series of steady-state configurations which correspond to various values of the parameter L. We now state the results for two models.

5. (a) *Point-source Model.*—For given M and κ, there exists a value L_1 such that for $L > L_1$ no steady-state configuration exists; the reason is that radiation-pressure is too high for equilibrium. There

* The standard model generalises at once with the familiar model "$\kappa\eta$ = constant" as shown by Eddington. The point-source model has a similar generalisation. Since "η" $\to$ 1 at the boundary, when "$\kappa\eta$" is constant the value of the constant is of course κ_1, the boundary opacity. To deny this, as in the current theory, is a sophistry. If it is not κ_1, then "$\kappa\eta$" is not constant, and a new investigation is required.

† This is the usual condition assumed, for example, in all Emden's and Eddington's models. If we wish we may take account of the more exact Schwarzschild-Hopf-Bronstein relation $T_e^4 = T_0^4(4/\sqrt{3})$, which with $L = 4\pi r_1^2(\frac{1}{4}ac)T_e^4$ provides a further relation for determining a further unknown T_0. But complications then ensue due to the tendency of the stellar atmosphere to extend to infinity. It seems best to avoid these complications at first and take T = 0 when ρ = 0.

also exists a value $L_0 (< L_1)$ such * that for $L = L_0$, $M(r) \to 0$ as $r \to 0$. It is then shown that for $L_1 > L > L_0$, $M(r) \to$ a positive limit $M(0)$ as $r \to 0$; for $L_0 > L > 0$, $M(r) \to$ a negative limit $M(0)$ as $r \to 0$.

Hence for $L_1 > L > L_0$ the gas-laws ultimately break down; for, since $M(r)$ is contained within r, the mean density of the matter enclosed within r, namely, $M(r)/\frac{4}{3}\pi r^3$, increases indefinitely as r decreases, *if the gas-laws continue to hold*. It follows that for $L_1 > L > L_0$ there must be a core of very high density, where the gas-laws fail. Outside the region of failure the perfect-gas solution $M(r)$ holds. For perfect gas of unlimited compressibility we have the limiting case of a point-mass $M(0)$ at the centre.

If $L_0 > L > 0$, $M(r) \to$ a negative limit $M(0)$, and no such interpretation is possible. But this case is just as significant as the previous one. Since $M(r) \to$ a negative limit as $r \to 0$, and since $M(r)$ varies continuously, there must be a first zero of $M(r)$ as r decreases from r_1 inwards. Let it occur at $r = r'$. Here $M(r') = 0$, but the pressure and density at $r = r'$ are found to be non-zero. Hence a perfect-gas steady-state configuration can be constructed as follows. At $r = r'$ construct an artificial transparent supporting surface, spherical in shape, containing no matter, but enclosing a central point-source L. Outside this deposit the mass $M(r_1) - M(r') = M$. Then this mass, thus supported, can exist in the form of a spherical shell of perfect gas, in a steady state, emitting L. We have thus constructed an artificially supported configuration satisfying all our conditions; and since $M(r)$ is bound to vanish at $r = r'$, no other perfect-gas configuration with the prescribed M and L ($L_0 > L > 0$) can exist. But actually the configuration is not provided with an artificial supporting surface. To see the actual form of the configuration for $L_0 > L > 0$ we therefore construct the artificially supported configuration and then remove the supporting surface. The mass must immediately collapse. And since no other perfect-gas configuration with this L and M exists, save the artificially supported one, the collapse must proceed until the gas-laws are violated. The mass cannot disappear. Hence it ultimately takes up a steady-state configuration in which at least the central region is not in a state of perfect gas. In the process of collapse the external radius decreases, and though the outer fringe may remain gaseous, the mean density will necessarily become very large.

We thus see that for $L = L_0$ a perfect-gas diffuse distribution is possible; for this model, it is found to have an arbitrary radius. But both for $L_1 > L > L_0$ and $L_0 > L > 0$ no perfect-gas configuration is possible. For $L_1 > L > L_0$ we have configurations with a dense central region surrounded by an extensive gaseous envelope; for $L_0 > L > 0$ we have almost the whole mass in a dense state surrounded by a gaseous fringe. It follows that the diffuse density-distribution $L = L_0$ is unstable. Let L increase above L_0 ever so slightly, and the new steady-state configuration possesses a central condensation; let L decrease ever so slightly below L_0, and a collapse must occur. Thus the configurations $L = L_0$ are bordered by configurations of totally different

* This follows from Eddington's investigations.

type, each type having enormously less gravitational potential energy ; hence in general the configurations $L = L_0$ are unstable to small changes of L. There will be one configuration $L = L_0$ which is the limit of the configurations for $L_0 > L > 0$ as $L \to L_0 - 0$, namely, that one of the series $L = L_0$ which has such a radius that the central density just reaches the limit of validity of the gas-laws.

Configurations of the type $L_1 > L > L_0$ will be called "centrally-condensed." Configurations of the type $L_0 > L > 0$ will be called "collapsed." The term is not to be taken to imply that they have necessarily undergone an actual collapse ; it simply refers to the method of constructing them.

It will be observed that the high density and the failure of the gas-laws in the core of a centrally-condensed configuration arise from an entirely different reason from that in the core of a collapsed configuration. For centrally-condensed configurations ($L_1 > L > L_0$) the failure arises from the inability of the material to be capable of infinite compressibility. But for collapsed configurations ($L_0 > L > 0$) the failure arises from the inability of a mass of perfect-gas to sustain its own weight when L, and therefore radiation-pressure, falls below a certain limit L_0. No perfect-gas configuration of equilibrium is possible when $L_0 > L > 0$, whatever the degree of compressibility. For $L_0 > L > 0$, imperfect compressibility is not the *origin* of the collapsed state ; it is simply the mechanism by which a steady state is ultimately achieved. The origin of the collapsed state is the inadequacy of perfect gas to sustain its own weight for $L_0 > L > 0$.

In simple English, *perfect-gas* configurations have a central condensation for $L_1 > L > L_0$, no condensation for $L = L_0$, and a hole or cavity artificially maintained for $L_0 > L > 0$. Hence perfect-gas configurations cannot exist for $L_0 > L > 0$, for the hole must tend to fill itself in, and we get "collapsed" configurations.

For the point-source model the series of perfect-gas centrally-condensed configurations ($L_1 > L > L_0$) does not pass continuously into one of the configurations $L = L_0$ as $L \to L_0 + 0$, for the central density changes discontinuously from ∞ to a finite value. Whether the passage is continuous when non-obedience to the classical gas-laws is taken into account I have not yet been able to discover.

(b) *Standard Model* (κ = constant, ϵ = constant).—When we carry through the same analysis for the standard model the results are as follows. For given M and κ, there exists a value L_1 such that for $L > L_1$ no steady-state configuration exists. There also exists a value L_0 with the following properties: For $L_1 > L > L_0$, $M(r) \to 0$ as $r \to 0$, but the density $\rho \to \infty$ as $r \to 0$. For $L = L_0$, $M(r) \to 0$, and $\rho \to$ a finite limit as $r \to 0$. For $L_0 > L > 0$, $M(r)$ has a first zero for some value $r = r'$ less than r_1. Hence when $L_1 > L > L_0$, there is a central condensation, where the classical gas-laws break down owing to the high density. When $L_0 > L > 0$, we can construct artificially a perfect-gas configuration of equilibrium, of the prescribed L and M, by introducing an artificial supporting surface of radius r' containing no matter. Outside this deposit the mass $M(r_1) - M(r') = M$. Then

this mass can exist in the form of perfect gas in a steady state, emitting L. But actual configurations are not provided with internal supporting surfaces. To find the nature of the actual configurations for $L_0 > L > 0$, construct an artificial one and then remove the supporting surface. The mass must collapse, and since no other perfect-gas configuration is possible save the artificially supported one, collapse will proceed until the gas-laws are disobeyed. We arrive at a " collapsed " configuration of high density.

Thus the results are the same for the two outlying cases, the point-source model and the standard model. Since nothing is involved except (*a*) the form of the classical gas-laws, (*b*) fundamental properties of solutions of different equations, we infer that the same result will hold for any source-distribution.* Thus for given M and κ we have the result that there exist two critical luminosities, L_1 and L_0 ($L_1 > L_0$), with the following properties: For $L > L_1$ no configuration exists; for $L_1 > L > L_0$ the configuration has a central condensation surrounded by a gaseous envelope; for $L = L_0$ the configurations are diffuse; for $L_0 > L > 0$ the configuration is a collapsed one. Save for $L = L_0$ the configurations have high central densities, and for $L_0 > L > 0$ high mean densities as well; for $L = L_0$ the configuration is in general unstable.

6. *Application to Nature. The Explanation of " White Dwarfs."*— This very striking classification of steady-state configurations appears to be the explanation of a fundamental bifurcation observed in nature, namely, that stars are either of " normal " density (ordinary gas, liquid and solid densities as known terrestrially) or of abnormally high density. The former are the class of giants and ordinary dwarfs, the latter are the " white dwarfs." We therefore identify centrally-condensed configurations with giants and ordinary dwarfs, collapsed configurations with white dwarfs. The explanation of the existence of very dense stars is simply that, for values of L below a certain critical value depending on M and κ, no perfect-gas configurations, even centrally condensed, are possible; light-pressure is too small.

The critical luminosity $L = L_0$ which separates the centrally-condensed stage from the collapsed stage can be calculated for any given mass, distribution of source-strength, and opacity by the methods of this paper. In this paper itself I only give the analysis for the point-source model and the standard model, both being for κ = constant. I make no attempt to evaluate L_0 for actual stars. It must be remarked that the evaluation of L_0 is a difficult business, as it requires an accurate knowledge of κ right up to the boundary of the star. To see this, consider a diffuse configuration $L = L_0$ with κ = constant, and suppose that in some particular spherical layer κ is altered. If κ is decreased, radiation-pressure is decreased, and the corresponding configuration for the same L must be a collapsed one; to recover the diffuse distribution a higher value of L_0 will be required. If κ is increased, radiation-pressure is increased, and the corresponding

* I have in hand the outline of a proof of the result for any source-distribution, but I am content at present to leave the argument as above.

configuration for the same L must be a centrally-condensed one ; to recover the diffuse distribution a lower value of L_0 will be required. This argument holds good in whatever layer κ is altered, not excluding the surface layers. In short, change of κ anywhere in a diffuse distribution has a devastating effect * on the density-distribution for given L.

The analysis shows how to calculate the densities, radii, etc., of " white dwarfs," and of giants and ordinary dwarfs, from purely *a priori* considerations, as functions of κ, L, and M.

The analysis discloses no reason why " white dwarfs " should be " white," and suggests, in fact, that for sufficiently small luminosities they should be red. This is a matter for future research and observation. As to why the ordinary stars should have a maximum effective temperature, and for temperatures below this should exhibit two ranges of luminosity, the analysis does indeed provide a hint. But this occurs at a much later stage of the analytical development, and involves the mass-luminosity law, *i.e.* something beyond equilibrium considerations.

I now proceed to the analysis.

(A) *The Point-source Model of Mass M, Luminosity L, and Constant Opacity.*†

7. *Equations of the Problem.*—The problem is to determine the configurations of equilibrium for given M, κ, and arbitrary L. Let p be the gas-pressure, $p'(= \frac{1}{3}aT^4)$ the radiation-pressure, T the temperature, ρ the density at distance r from the centre. Then the well-known equations are

$$\frac{dp}{dr} + \frac{dp'}{dr} = -\frac{GM(r)}{r^2}\rho \quad . \quad . \quad . \quad (1)$$

$$\frac{dp'}{dr} = -\frac{\kappa L}{4\pi c r^2}\rho \quad . \quad . \quad . \quad . \quad (2)‡$$

$$M(r) = M(r_2) + \int_{r_2}^{r} 4\pi\rho r^2 dr \quad . \quad . \quad . \quad (3)$$

* The critical luminosity L_0 would be the theoretical luminosity *predicted* for stars by the current theory did the current theory carry through its calculations free from mistakes. The current theory gives L_0 correctly for the standard model, for in that case the current theory carries the integration of the equation to the boundary. In all other cases, as it is a " feature " of the current theory that it " dispenses with any detailed discussion of the outer $\frac{1}{10}$ of the radius " (Eddington's words, *M.N.*, **90**, 809, 1930), the current theory fails to ascertain L_0 correctly. Thus, in excluding from consideration the outer layers of a star, the current theory makes a doubly fatal error: it debars itself from ascertaining the true structure of a star by inward integration, and it fails even to ascertain correctly the critical luminosity L_0 of diffuse configurations.

† My investigation of the point-source model is on substantially different lines from Mr. T. G. Cowling's, printed in this number of *M.N.* The fundamental idea in my investigation is first the forming of sufficient boundary conditions to begin the integration at the boundary and work inwards, and then the reduction to dependence on a single parameter.

‡ That this equation holds up to the boundary was shown in my Bakerian Lecture, *Phil. Trans.*, **228** A, 451, 1929, equations (66) and (67).

where $M(r)$ is the mass inside radius r and $M(r_2)$ that inside r_2, r_2 being arbitrary. Change to variables ξ, θ, χ by the relations

$$r = \lambda_1\xi, \qquad T = \lambda_2\theta, \qquad \rho = \lambda_3\chi \quad . \quad . \quad (4)$$

where λ_1, λ_2, λ_3 are coefficients to be determined. So long as the classical gas-laws persist we have

$$p = (R/\mu)\rho T,$$

where μ is the molecular weight, assumed constant. Substitute (4) in (1) and choose λ_2, λ_3 so that

$$(R/\mu)\lambda_3 = \tfrac{1}{3}a\lambda_2^3 \quad . \quad . \quad . \quad . \quad (5)$$

Then (1) becomes

$$\frac{R}{\mu}\lambda_1\lambda_2\frac{d}{d\xi}(\theta\chi + \theta^4) = -\frac{\chi}{\xi^2}GM(r) \quad . \quad . \quad (6)$$

and (2) becomes

$$\frac{R}{\mu}\lambda_1\lambda_2\frac{d(\theta^4)}{d\xi} = -\frac{\kappa L}{4\pi c}\frac{\chi}{\xi^2} \quad . \quad . \quad . \quad (7)$$

Now choose λ_1 and λ_2 so that

$$\frac{R}{\mu}\lambda_1\lambda_2 = \frac{\kappa L}{4\pi c} \quad . \quad . \quad . \quad . \quad (8)$$

Then (6) becomes

$$\frac{d}{d\xi}(\theta\chi + \theta^4) = -\frac{\chi}{\xi^2}\frac{4\pi cGM(r)}{\kappa L} \quad . \quad . \quad . \quad (9)$$

and (7) becomes

$$\frac{d(\theta^4)}{d\xi} = -\frac{\chi}{\xi^2} \quad . \quad . \quad . \quad . \quad (10)$$

Now put

$$\phi = \frac{4\pi cGM(r)}{\kappa L} \quad . \quad . \quad . \quad . \quad (11)$$

Then (9) becomes

$$\frac{d(\theta\chi + \theta^4)}{d\xi} = -\frac{\chi\phi}{\xi^2},$$

or using (10),

$$\frac{d(\theta\chi)}{d\xi} = -\frac{\chi(\phi - 1)}{\xi^2} \quad . \quad . \quad . \quad (12)$$

Again, by (3) and (11),

$$\frac{d\phi}{dr} = \frac{4\pi cG}{\kappa L}4\pi\rho r^2.$$

In this, substitute for r and ρ in terms of ξ and χ, and eliminate λ_2 and λ_3 by means of (5) and (8). We find that λ_1 disappears and we have finally

$$\frac{d\phi}{d\xi} = \frac{4\pi G(\frac{1}{3}a)}{(R/\mu)^4}\left(\frac{\kappa L}{4\pi c}\right)^2\chi\xi^2 \quad . \quad . \quad . \quad (13)$$

The initial values of θ and χ are zero, if we adopt the usual condition that T and ρ vanish at the boundary. The constant λ_1 is still at our

disposal; choose it so that $\xi = 1$ at the boundary, *i.e.* $\lambda_1 = r_1$, where r_1 is the radius. The boundary value of ϕ, say ϕ_1, is $4\pi c\mathrm{GM}/\kappa\mathrm{L}$ by (11). Define a number β by the equation

$$\beta = 1 - \frac{\kappa \mathrm{L}}{4\pi c \mathrm{GM}} \quad . \quad . \quad . \quad . \quad (14)$$

Then

$$\phi_1 = \frac{1}{1-\beta}.$$

Thus the problem is to solve the differential equations (10), (12), and (13) with the boundary conditions that at $\xi = 1$, $\theta = 0$, $\chi = 0$, $\phi = 1/(1-\beta)$.

It is convenient now to change to new variables θ', χ', ϕ' given by

$$\phi - 1 = \frac{\beta}{1-\beta}\phi', \qquad \theta = \frac{\beta}{1-\beta}\theta', \qquad \chi = \left(\frac{\beta}{1-\beta}\right)^4 \chi' \quad . \quad (15)$$

Then equations (10), (12), (13) become

$$\frac{d(\theta'^4)}{d\xi} = -\frac{\chi'}{\xi^2} \quad . \quad . \quad . \quad . \quad (16)$$

$$\frac{d(\theta'\chi')}{d\xi} = -\frac{\chi'\phi'}{\xi^2} \quad . \quad . \quad . \quad . \quad (16')$$

$$\frac{d\phi'}{d\xi} = \frac{4\pi(\frac{1}{3}a)\mathrm{G}^3\mathrm{M}^2}{(\mathrm{R}/\mu)^4}\frac{\beta^3}{1-\beta}\chi'\xi^2 \quad . \quad . \quad . \quad (16'')$$

and the initial conditions are that at $\xi = 1$,

$$\theta' = 0, \qquad \chi' = 0, \qquad \phi' = 1.$$

8. *Solution.*—It is easily found that equations (16), (16′), (16″) with their boundary conditions have a solution regular at $\xi = 1$, expansible in power series in $(1-\xi)$. Since the boundary conditions are now independent of the constants of the problem, the solution depends on a single parameter,* the coefficient of the right-hand side of (16″). If we call this 1/C we have

$$\mathrm{C} = \frac{(\mathrm{R}/\mu)^4}{4\pi(\frac{1}{3}a)\mathrm{G}^3\mathrm{M}^2}\frac{1-\beta}{\beta^3} = \frac{(\mathrm{R}/\mu)^4\kappa\mathrm{L}}{(4\pi)^2\frac{1}{3}ac\mathrm{G}^4\mathrm{M}^3}\left(1-\frac{\kappa\mathrm{L}}{4\pi c\mathrm{GM}}\right)^{-3} \quad (17)$$

This depends only on the data of the problem and is known. C may be called the " discriminant " of the point-source model. The solution is therefore of the type

$$\theta' = f_1(\xi, \mathrm{C}),$$
$$\chi' = f_2(\xi, \mathrm{C}),$$
$$\phi' = f_3(\xi, \mathrm{C}),$$

* It is of course simple to transform equations (16), (16′), (16″) so as to free them from the constants. The constants then appear in the boundary conditions. The results are unaffected.

and T, ρ, and M(r) are then found to be given by

$$T = \frac{GM}{r_1}\frac{\mu}{R}\left(1 - \frac{\kappa L}{4\pi cGM}\right) f_1(\xi, C) \quad . \quad . \quad (18)$$

$$\rho = \frac{\frac{1}{3}a}{(R/\mu)^4}\frac{G^4M^4}{{r_1}^3}\frac{4\pi c}{\kappa L}\left(1 - \frac{\kappa L}{4\pi cGM}\right)^4 f_2(\xi, C) \quad . \quad (18')$$

$$M(r) = \frac{\kappa L}{4\pi cG} + \left(M - \frac{\kappa L}{4\pi cG}\right) f_3(\xi, C) \quad . \quad . \quad (18'')$$

and

$$r = r_1\xi \quad . \quad . \quad . \quad . \quad . \quad (18''')$$

These equations are valid, as we go inwards, so long as the gas-laws hold. But the functions f_1, f_2, f_3 exist *analytically* the whole way from $\xi = 1$ to $\xi = 0$.

9. *Deductions from the Solution.*—It is clearly necessary that

$$\frac{\kappa L}{4\pi cGM} < 1,$$

for the pressure ($\propto \theta\chi$) must increase inwards, and so ϕ must be initially greater than unity. If $L_1 = 4\pi cGM/\kappa$, we have therefore $L < L_1$.

From the researches of Eddington on point-source models, and from T. G. Cowling's paper (see p. 92), we know that there exists a value of L for which $M(r) \to 0$ as $r \to 0$. Call this L_0. Then by (18″) L_0 is a root * of the equation

$$-\frac{\kappa L}{4\pi cGM - \kappa L} = f_3(0, C) \quad . \quad . \quad . \quad (19)$$

where of course C depends on L, M, κ. This root L_0 is accordingly a function of M and κ. Now it is clear from (16″) that $f_3(\xi, C)$ decreases from its initial value, unity, the more rapidly the smaller the value of C, *i.e.* the smaller is L ; and therefore when $L < L_0$, before ξ reaches 0, ξ will have reached a value ξ' at which

$$-\frac{\kappa L}{4\pi cGM - \kappa L} = f_3(\xi', C).$$

For $f_3(\xi, C)$ has only to decrease to $-\kappa L/(4\pi cGM - \kappa L)$, which is algebraically larger than $-\kappa L_0/(4\pi cG - \kappa L_0)$, (since $L < L_0$) and so reaches it sooner than $\xi = 0$. At this place $\xi = \xi'$, $M(r') = 0$. On the other hand, when $L > L_0$, $f_3(\xi, C)$ will have decreased less at $\xi = 0$ than when $L = L_0$, and so accordingly $M(0) > 0$ for $L > L_0$.

Thus, as described in paragraph 5, for $L_1 > L > L_0$ we have a positive central point-mass if we continue the solution to the centre, and so actually a centrally-condensed configuration with high central density. For $L_0 > L > 0$ we can only secure a steady-state perfect-gas configuration by inserting an artificial supporting surface at $r = r'$, containing no mass but only the point-source L. In the absence of a

* The current theory *assigns* this luminosity L_0 to the star. But clearly there is no reason whatever why the star should have this particular luminosity.

supporting surface, the configuration for $L_0 > L > 0$ must accordingly be a collapsed one.

It will be seen that the whole character of the configuration depends upon two pure numbers, β and C (calculable in terms of L, M, and κ), according to the inequalities $L \gtrless L_0$, or

$$\frac{1-\beta}{\beta} + f_3(0, C) \gtrless 0.$$

(B) *The Standard Model, κ = Constant, ϵ = Constant. (L and M given.)*

10. *Equations of the Problem.*—The problem is to determine the configurations of equilibrium for given M and κ, for arbitrary L, given ϵ = constant. Let $L(r)$ be the source-strength inside radius r. The equations are

$$\frac{dp}{dr} + \frac{dp'}{dr} = -\frac{GM(r)}{r^2}\rho \quad . \quad . \quad . \quad (20)$$

$$\frac{dp'}{dr} = -\frac{\kappa L(r)}{4\pi c r^2}\rho \quad . \quad . \quad . \quad (21)$$

and $$\frac{L(r)}{M(r)} = \epsilon = \frac{L}{M}.$$

Hence $$1 + \frac{dp}{dp'} = \frac{4\pi c GM}{\kappa L}.$$

If $p = 0$ when $p' = 0$, we have on integration *

$$p = p'\left(\frac{4\pi c GM}{\kappa L} - 1\right) \quad . \quad . \quad . \quad (22)$$

This holds whatever the equation of state. Clearly it is necessary that

$$\frac{4\pi c GM}{\kappa L} > 1$$

or $$L < L_1,$$

where $$L_1 = 4\pi c GM/\kappa.$$

Put † $$\beta = 1 - \frac{\kappa L}{4\pi c GM} \quad . \quad . \quad . \quad . \quad (23)$$

---

* It is the existence of this integral which makes the standard model so much easier to treat in detail than the point-source model. Instead of three first-order equations we have ultimately a single second-order equation.

† In all the formulæ of this paper β is equal to $1 - \frac{L}{L_1}$. If $1 - \frac{L}{L_1}$ is substituted for β in all the formulæ, they then contain no explicit reference to opacity, and they may be expected to give insight into the nature of the configurations for other distributions of source and opacity. The critical value L_1 will of course depend on the law of opacity introduced.

so that β is known in terms of the data L, M, and κ. Then

$$p = p'\frac{\beta}{1-\beta} \quad . \quad . \quad . \quad . \quad (22')$$

11. We now investigate how far inwards we can go with steady-state configurations on the standard model composed of perfect gas. Introducing the equation of state

$$p = (\mathrm{R}/\mu)\rho\mathrm{T},$$

(22′) gives

$$(\mathrm{R}/\mu)\rho = \tfrac{1}{3}a\mathrm{T}^3\frac{\beta}{1-\beta} \quad . \quad . \quad . \quad (24)$$

Substituting from (22′) and (24) in (20) we find

$$4\frac{\mathrm{R}}{\mu\beta}r^2\frac{d\mathrm{T}}{dr} = -\mathrm{GM}(r) \quad . \quad . \quad . \quad (25)$$

Now

$$\frac{d\mathrm{M}(r)}{dr} = 4\pi r^2\rho.$$

Differentiating (25) and substituting for ρ from (24) we find

$$\frac{1}{\frac{1}{3}\pi a\mathrm{G}}\frac{1-\beta}{\beta^2}\left(\frac{\mathrm{R}}{\mu}\right)^2\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\mathrm{T}}{dr}\right) = -\mathrm{T}^3 \quad . \quad . \quad (26)$$

Put

$$\mathrm{T} = \lambda\theta, \qquad r = \frac{\xi}{\lambda}\frac{\mathrm{R}}{\mu}\frac{(1-\beta)^{\frac{1}{2}}}{\beta}\frac{1}{(\frac{1}{3}\pi a\mathrm{G})^{\frac{1}{2}}} \quad . \quad . \quad (27)$$

where λ is arbitrary. Then (26) clears of constants and becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3 \, . \quad . \quad . \quad . \quad (28)$$

which is Emden's polytropic equation "$n = 3$."

12. *The Discriminant Equation.*—In (25), which is merely a form of the equation of mechanical equilibrium, substitute for T and r from (27).* It becomes

$$\mathrm{M}(r) = -\frac{4}{(\frac{1}{3}\pi a\mathrm{G}^3)^{\frac{1}{2}}}\frac{(1-\beta)^{\frac{1}{2}}}{\beta^2}\left(\frac{\mathrm{R}}{\mu}\right)^2\xi^2\frac{d\theta}{d\xi} \quad . \quad . \quad (29)$$

Apply (29) to the boundary of the star, letting $r \to$ its boundary value r_1, $\mathrm{M}(r) \to \mathrm{M}$, $\xi \to \xi_1$. We have

$$\mathrm{M} = -\frac{4}{(\frac{1}{3}\pi a\mathrm{G}^3)^{\frac{1}{2}}}\frac{(1-\beta)^{\frac{1}{2}}}{\beta^2}\left(\frac{\mathrm{R}}{\mu}\right)^2\xi_1^2\left(\frac{d\theta}{d\xi}\right)_1 \quad . \quad . \quad (30)$$

Put †

$$\mathrm{C} = \frac{1-\beta}{\beta^4}\left(\frac{\mathrm{R}}{\mu}\right)^4\frac{16}{\frac{1}{3}\pi a\mathrm{G}^3\mathrm{M}^2} \quad . \quad . \quad . \quad (31)$$

---

* This use of equation (25) is the characteristic step of the present method.

† Apart from a factor 4 this is the definition of C given in my earlier paper when applied to the standard model (*M.N.*, **90**, 19, 1929, equations (24) and (25)). Eddington, selecting Emden's special solution of Emden's polytropic equation, attributes the value (2·0150) [2] to C, with which value (31) becomes Eddington's quartic equation with his numerical value of the coefficient of $\mathrm{M}^2\beta^4\mu^4$. From this he computes a luminosity L, the root of (31′). But of course there is no reason

or, substituting for β,

$$C = \frac{4\kappa L}{\pi^2(\frac{1}{3}a)cG^4M^3}\left(\frac{R}{\mu}\right)^4\left(1 - \frac{\kappa L}{4\pi cGM}\right)^{-4} \quad . \quad . \quad (31')$$

C is thus a function of L, M, and κ.

Then by (30),

$$\xi_1^2\left(\frac{d\theta}{d\xi}\right)_1 = -\frac{1}{C^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (32)$$

I propose to call C the "discriminant" of the standard model, and equation (30) the "discriminant equation." For given M and κ, C decreases as L decreases, and accordingly $\xi_1^2\,|\,d\theta/d\xi\,|_1$ increases as L decreases.

13. Choose the arbitrary constant λ so that $\xi_1 = 1$, *i.e.* by (27) choose

$$\lambda = \frac{1}{r_1}\frac{R}{\mu}\frac{(1-\beta)^{\frac{1}{2}}}{\beta}\frac{1}{(\frac{1}{3}\pi a G)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (33)$$

Then the problem is to integrate Emden's polytropic equation (28),

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3 \quad . \quad . \quad . \quad . \quad (34)$$

with the initial conditions that

$$\text{at} \quad \xi = \xi_1 = 1, \qquad \theta = 0, \qquad \text{and} \quad \left(\frac{d\theta}{d\xi}\right)_1 = -\frac{1}{C^{\frac{1}{2}}} \quad . \quad (35)$$

where C is given in terms of L, M, and κ by (31').

We shall show that for different values of C, the integral curves of equation (28) passing through $\xi = 1$, $\theta = 0$, are of the form shown in fig. 1. From this diagram we shall be able to determine the steady-state perfect-gas configurations. These are determined entirely by the starting value of $d\theta/d\xi$, *i.e.* by the value of C. As I stated in my earlier paper, C "recapitulates in itself those aspects of the density-distribution which are significant for the surface-cooling; it is a telescoped version of the structure of the star between centre and boundary." *

14. *Discussion of the Integrals.*—In the first place there is one, and only one, integral curve through $\xi = 1$, $\theta = 0$, remaining positive in the interval $1 > \xi > 0$, and running to a finite value of θ (say θ_0) at $\xi = 0$; and for this curve $d\theta/d\xi = 0$ at $\xi = 0$.

For, put $\theta = y/\xi$. Emden's equation (28) becomes

$$\xi^2\frac{d^2y}{d\xi^2} = -y^3 \quad . \quad . \quad . \quad . \quad (36)$$

why L should have this value. From its mode of derivation (30) is a boundary relation. Equation (30) was first given by me in *M.N.*, **83**, 128, 1923, equation (29), "The Equilibrium of a Rotating Star." Eddington's "quartic equation" is thus seen to be a boundary relation, though this fact is concealed in his deduction of it from dimensional considerations. The substantial equivalence of what are here called "boundary relations" to photospheric relations in actual stars was demonstrated in my earlier paper, *M.N.*, **90**, 32, 1929, § 20.

* *Loc. cit.*, p. 5.

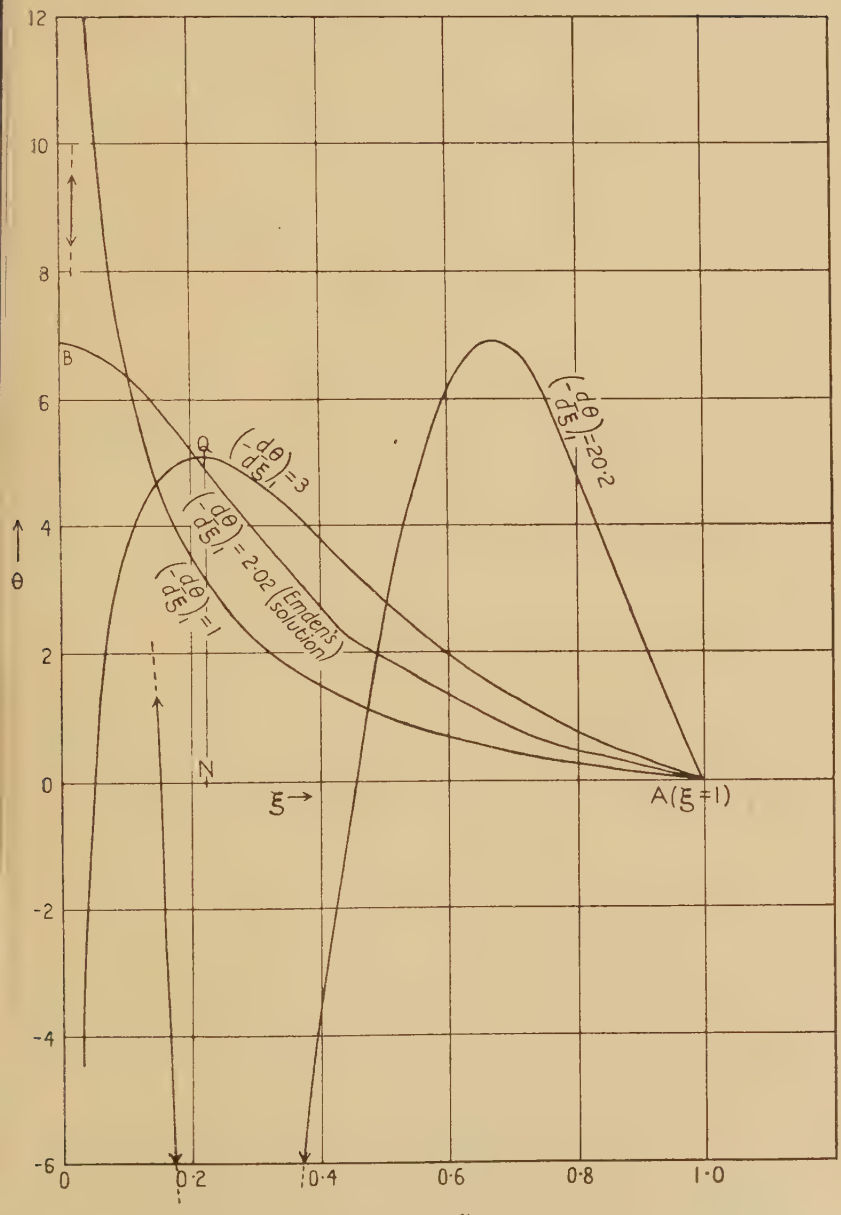

FIG. 1.—Graphs of solutions of $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3$ through $\xi = 1$, $\theta = 0$.

If there is an integral for which $\theta \to$ a finite value θ_0 as $\xi \to 0$, then along this integral *

$$\lim_{\xi\to 0} \frac{d\theta}{d\xi} = \lim_{\xi\to 0} \frac{\xi(dy/d\xi) - y}{\xi^2} = \lim_{\xi\to 0} \frac{1}{2}\frac{d^2y}{d\xi^2} = -\lim_{\xi\to 0} \frac{1}{2}\frac{y^3}{\xi^2} = -\lim_{\xi\to 0} \frac{1}{2}\xi\theta^3 = 0.$$

Now there is at least one such integral. For we can construct the solution of (28) (Emden's solution) for which

$$\text{at} \quad \xi = 0, \qquad \theta = 1, \quad d\theta/d\xi = 0,$$

obtaining an integral $\theta = f(\xi)$. This is the well-known Emden polytrope "$n = 3$," which, according to Emden's table, possesses a zero at

$$\xi = \xi_0 = 6{\cdot}9011,$$

at which point

$$f'(\xi_0) = -0{\cdot}04231, \qquad \xi_0^2 f'(\xi_0) = -2{\cdot}0150.$$

But if $\theta = \mathrm{F}(\xi)$ is *any* solution of (28), $\theta = \mathrm{AF}(\mathrm{A}\xi)$ is also a solution where A is arbitrary. Choose $\mathrm{A} = \xi_0$. Then $\theta = \xi_0 f(\xi_0\xi)$ is a solution vanishing for $\xi = 1$ and meeting $\xi = 0$, where $\theta = \xi_0$ and $d\theta/d\xi = 0$. The same argument shows that there is only one such integral everywhere positive in $0 < \xi < 1$. For if there were two, say $\theta = f_1(\xi)$ and $\theta = f_2(\xi)$, meeting $\xi = 0$ where $\theta = \theta_0^{(1)}$ and $\theta = \theta_0^{(2)}$ respectively, and having $\xi = 1$ for their first zero as ξ increases from 0, then

$$\theta = \frac{\theta_0^{(1)}}{\theta_0^{(2)}} f_2\left(\frac{\theta_0^{(1)}}{\theta_0^{(2)}}\xi\right)$$

would also be a solution, and it meets $\xi = 0$ where $\theta = \theta_0^{(1)}$ and $d\theta/d\xi = 0$. Thus there would be two solutions, namely, the one just mentioned and $\theta = f_1(\xi)$, having common values of θ and $d\theta/d\xi$ at $\xi = 0$. By the theory of differential equations these must be identical, and so, since $\xi = 1$ is the first zero of each, $f_1 \equiv f_2$, $\theta_0^{(1)} = \theta_0^{(2)}$.

The unique positive integral † through $\xi = 1$, $\theta = 0$ which reaches $\xi = 0$ at a finite value of θ, namely, $\theta = \xi_0 f(\xi_0\xi)$, will have a definite slope at $\xi = 1$, given by

$$\left(\frac{d\theta}{d\xi}\right)_1 = \frac{d}{d\xi}[\xi_0 f(\xi_0\xi)]_{\xi=1} = \xi_0^2 f'(\xi_0) = -2{\cdot}0150.$$

Put ω for this number 2·0150 and let $\mathrm{C}_0 (= 1/\omega^2)$ be the corresponding value of C according to (32). Let L_0 be the corresponding value of L according to (31′) for any given M and κ. L_0 is thus a function of M and κ, given by the equation

$$\frac{1}{\omega^2} = \frac{4\kappa \mathrm{L}_0}{\pi^2(\frac{1}{3}a)c\mathrm{G}^4\mathrm{M}^3}\left(\frac{\mathrm{R}}{\mu}\right)^4\left(1 - \frac{\kappa \mathrm{L}_0}{4\pi c\mathrm{GM}}\right)^{-4}.$$

* The argument is due to Emden, *Gaskugeln*, p. 199 (1907).

† There is, of course, the reflection of this integral in the axis of ξ as well. But this makes θ negative in (0, 1). Similarly there are other integrals for which θ has earlier zeros in (0, 1) than $\xi = 1$. These arise from the analytical continuation of Emden's solution past its first zero.

Then if L happens to have the exact value L_0, the distributions of temperature T ($\propto \theta$) and density ρ ($\propto \theta^3$) are given by the Emden curve AB of the diagram. But of course there is no reason why L should have the value L_0. (The current theory *attributes* the luminosity L_0 to the star.)

We next consider what happens when L is not equal to L_0. In this case C does not take the value C_0, $(d\theta/d\xi)_1$ is not equal to $-\omega$, and θ does not tend to a finite limit as $\xi \to 0$. It can only tend to $\pm\infty$ or oscillate infinitely.* I determine the behaviour near $\xi = 0$ as follows. In (36) write $\log(1/\xi) = x$. The equation becomes

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = -y^3 \quad . \quad . \quad . \quad . \quad (37)$$

In (37) put $x^{-\frac{1}{2}} = t$. The equation becomes

$$\frac{1}{4}t^6\frac{d^2y}{dt^2} + \frac{3}{4}t^5\frac{dy}{dt} - \frac{1}{2}t^3\frac{dy}{dt} = -y^3 \quad . \quad . \quad . \quad (38)$$

As $\xi \to 0$, $x \to \infty$, and $t \to 0$. The last term in the left-hand side of (38) is dominant, and the asymptotic solution near $t = 0$ is seen by integration of $-\frac{1}{2}t^3dy/dt = -y^3$ to be

$$y \sim \frac{\pm 1}{(2t^{-2} + \text{const})^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad (39)$$

or, returning to the original variables,

$$\theta \sim \pm \frac{1/\sqrt{2}}{\xi\,(\log D/\xi)^{\frac{1}{2}}} \qquad (\xi \to 0) \quad . \quad . \quad . \quad (40)$$

where D is some constant. The later terms in the expansion of which (40) is the leading term will involve a second constant, which may be taken to be $(d\theta/d\xi)_1$. D is fixed by the condition that the analytical continuation of (40) passes through $\xi = 1$, $\theta = 0$. Since this analytical continuation depends on $(d\theta/d\xi)_1$ as well as on D, there will be a relation between D and $(d\theta/d\xi)_1$, *i.e.* D is of the form of a function of C, and so of L, M, κ.

It follows from (40) that some integral curves tend to $+\infty$ as $\xi \to 0$, others to $-\infty$ as $\xi \to 0$. Since the integral curve for which $C = C_0$ does not tend to $\pm\infty$, it follows that Emden's solution separates a pencil of solutions through $\xi = 1$, $\theta = 0$, which go to $+\infty$ as $\xi \to 0$, from a pencil which go to $-\infty$ as $\xi \to 0$. The latter must have a zero in the interval $0 < \xi < 1$. Thus Emden's solution ($(d\theta/d\xi)_1 = -\omega$, $C = C_0$, $L = L_0$) divides integral curves which pass to $+\infty$ as $\xi \to 0$ from integral curves which possess a zero in $0 < \xi < 1$. The question is, How do the integral curves lie with respect to Emden's curve? Do those which pass to $+\infty$ start with values of $|\,d\theta/d\xi\,|_1 > \omega$ or with values of $|\,d\theta/d\xi\,|_1 < \omega$?

At this stage of my investigations I requested Mr. N. Fairclough to carry out some numerical solutions of (34), taking definite initial values of $|\,d\theta/d\xi\,|_1$ not equal to ω. His first integration, for

* It has been shown by R. H. Fowler that it does not oscillate infinitely.

$|\,d\theta/d\xi\,|_1 = 10\omega$, gave a curve which, after oscillating three times across the axis of ξ, passed to $-\infty$. This strongly suggested that curves for which $|\,d\theta/d\xi\,|_1 < \omega$ pass to $+\infty$, curves for which $|\,d\theta/d\xi\,|_1 > \omega$ have at least one zero in $0 < \xi < 1$. Considerations suggesting a similar disposition of the curves were sent me the same day by Mr. T. G. Cowling. I rapidly performed myself two numerical investigations, namely, for $|\,d\theta/d\xi\,|_1 = 1$ and $|\,d\theta/d\xi\,|_1 = 3$, which confirmed this. The behaviour of the integral curves has since been exhaustively discussed by Mr. R. H. Fowler, who generously devoted his time, knowledge, and previous experience to this difficult problem. His results (see p. 63) confirming the above conjectures may be stated as follows :—

Behaviour of solutions of $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3$ *which begin at* $\xi = 1$, $\theta = 0$, *and have a negative value of* $d\theta/d\xi$ *at* $\xi = 1$, $\theta = 0$.

Case 1. $\left|\frac{d\theta}{d\xi}\right|_1 < \omega$, $(L_1 > L > L_0)$. $\theta \to +\infty$ as $\xi \to 0$, and θ is everywhere positive in $0 \leqslant \xi < 1$. As $\xi \to 0$,

$$\theta \sim \frac{1/\sqrt{2}}{\xi\,(\log D/\xi)^{\frac{1}{2}}}.$$

Case 2. $\left|\frac{d\theta}{d\xi}\right|_1 = \omega$, $(L = L_0)$. θ is everywhere positive and finite in $0 < \xi < 1$, and is represented by the Emden-Eddington solution.

Case 3. $\left|\frac{d\theta}{d\xi}\right|_1 > \omega$, $(L_0 > L > 0)$. θ has a first zero ξ' in $1 > \xi > 0$ as ξ decreases from unity, and in $1 > \xi > \xi'$, θ is positive and possesses a single maximum.*

These are the fundamental properties we require. We proceed to their astrophysical interpretation.†

15. *Case* 1. $L_1 > L > L_0$. The variable θ, and accordingly T and ρ, remains everywhere positive in $0 < \xi < 1$. Near $r = 0$, returning to the variables T and r, we have

$$T \sim \frac{1}{\sqrt{2}}\,\frac{1}{(\frac{1}{3}\pi a G)^{\frac{1}{2}}}\,\frac{R}{\mu}\,\frac{(1-\beta)^{\frac{1}{2}}}{\beta}\,\frac{1}{r\,(\log D'/r)^{\frac{1}{2}}} \quad . \quad . \quad (41)$$

$$\rho \sim \frac{1}{2\sqrt{2}}\,\frac{\frac{1}{3}a}{(\frac{1}{3}\pi a G)^{\frac{3}{2}}}\left(\frac{R}{\mu}\right)^2\frac{(1-\beta)^{\frac{1}{2}}}{\beta^2}\,\frac{1}{r^3\,(\log D'/r)^{\frac{3}{2}}} \quad . \quad (42)$$

where D′ is a constant depending on L, M, κ, r_1. We observe that T and $\rho \to \infty$ as $r \to 0$. It follows that near $r = 0$,

$$4\pi\int_0^r \rho r^2 dr \propto \int_0 \frac{r\,dr}{r\,(\log D'/r)^{\frac{3}{2}}} \propto \frac{1}{(\log D'/r)^{\frac{1}{2}}} \to 0$$

* In Case 3, θ will ultimately pass to $-\infty$ or to $+\infty$, as $\xi \to 0$, according as the number of zeros in (0, 1) is odd or even.

† Mr. Fowler's investigations also provide for the case where the integration is started a little way inside the star instead of at the fictitious boundary $\theta = 0$.

as $r \to 0$. Thus the mass given by the distribution of density is finite though the density becomes infinite. Further, the total mass M_1 given by the density distribution is

$$M_1 = \int_0^{r_1} 4\pi\rho r^2 dr = 4\pi \frac{\frac{1}{3}a}{R/\mu} \frac{\beta}{1-\beta} \lambda^3 \left[\frac{R/\mu}{\lambda} \frac{(1-\beta)^{\frac{1}{2}}}{\beta} \frac{1}{(\frac{1}{3}\pi a G)^{\frac{1}{2}}}\right]^3 \int_0^{\xi_1} \theta^3 \xi^2 d\xi$$

$$= \frac{4}{(\frac{1}{3}\pi a G^3)^{\frac{1}{2}}} \left(\frac{R}{\mu}\right)^2 \frac{(1-\beta)^{\frac{1}{2}}}{\beta^2} \left[-\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_1\right]$$

$$= M,$$

by (30), noting that $\int \theta^3 \xi^2 d\xi$ converges near $\xi = 0$. Hence our solution reproduces the desired mass M without a point-mass.

We have thus determined the steady-state configuration of an arbitrary mass M of arbitrary luminosity L lying in the range $L_1 > L > L_0$, when composed of ideally perfect gas of unlimited compressibility. The density and temperature become indefinitely large as we approach the centre ; the external radius is arbitrary.

In nature gases cease to obey the law $p = (R/\mu)\rho T$ at high densities. This in no way impairs the significance of our result. When $L_1 > L > L_0$, as we proceed inwards we necessarily continue with the solution we have found until the density becomes too large for the gas-laws to be obeyed ; *i.e.* there is necessarily a dense core. In this dense core T also becomes very large. We shall speak of such a configuration as being of " centrally-condensed " type.

Case 3. $L_0 > L > 0$. In this case θ becomes negative, oscillating across $\theta = 0$ one or more times before tending to $\pm \infty$. Negative densities having no meaning, it is clear that no steady-state configurations occupying the whole sphere $0 < \xi < 1$ exist when $L_0 > L > 0$. Nevertheless the solution is of profound astrophysical interest. Consider the point Q (fig. 1) where θ *first* attains a maximum. Call this point $r = r'$, $\xi = \xi'$. Here $(d\theta/d\xi)_{\xi=\xi'} = 0$, and hence by (29) $M(r') = 0$, and accordingly, too, $L(r') = 0$. In the shell $1 > \xi > \xi'$, ρ is always positive, and the total mass is

$$M(r_1) - M(r') = M.$$

Further, at $r = r'$, $\rho \neq 0$, and $T \neq 0$, for $\theta \neq 0$. Consequently we can construct a steady-state configuration as follows.

Take an artificial rigid supporting surface of totally reflecting * material of radius r' and let it enclose no mass. Deposit upon it the prescribed mass M of perfect gas. Then *the prescribed mass* M *of the prescribed luminosity* L (*lying in the range* $L_0 > L > 0$) *can exist in the form of a spherical shell of perfect gas, internally supported by the surface of radius* r'.

In nature no internal supporting surface is available. What then is the configuration of equilibrium ? To ascertain the nature of this,

* A transparent surface will serve equally well. For problems associated with the transmission of radiation across a hollow shell, see my paper, "The Radiative Equilibrium of a Planetary Nebula," *Zeits. für Astrophys.*, **1**, 98, 1930.

we first arrange the given mass in the perfect-gas form as a hollow shell internally supported, and then simply remove the supporting surface. The mass must collapse on to its own centre. We already know that no perfect-gas configuration exists other than the hollow shell internally supported. Consequently, the state of aggregation in the "collapsed" configuration cannot be that of a perfect gas. In nature the collapse will proceed, and the density increase, until the gas ceases to obey the classical gas-laws. A steady state will ultimately be reached since the material cannot disappear. Such a configuration will be called "collapsed," as opposed to the centrally-condensed type $L_1 > L > L_0$ and the diffuse type $L = L_0$.

Case 2. $L = L_0$. The diffuse configuration $L = L_0$, corresponding to the Emden-Eddington solution of Emden's differential equation, is essentially unstable. Let L increase above L_0 ever so slightly; then the corresponding new steady-state configuration possesses a central condensation. Let L decrease below L_0 ever so slightly; then the corresponding new configuration is a collapsed one. The configuration $L = L_0$ is bordered on either side by configurations of incomparably less gravitational potential energy, and so on ordinary statical principles is unstable.*

The diffuse configuration $L = L_0$ is also unstable to slight changes in the surface conditions. Throughout this investigation for the sake of simplicity I have taken $\kappa =$ constant. If κ takes a different value near the boundary, it is clear that the differential equation in the interior is substantially unaltered, but which solution of this differential equation, passing through or near $\xi = 1, \theta = 0$, is required remains to be determined. A glance at fig. 1 shows that a small disturbance in surface conditions, say in surface opacity, will move the solution from the Eddington-Emden curve to one of the neighbouring solutions tending to $+\infty$ as $\xi \to 0$, or to one of the neighbouring solutions which vanishes between $\xi = 1$ and $\xi = 0$, *i.e.* a slight change in surface conditions will transform the diffuse configuration into either a centrally-condensed configuration or a collapsed configuration. The slightest speck of meteoric dust falling on a star built on Eddington's model would cause it either to collapse or to develop a central condensation. In his recent discussion of a similar problem, Eddington *assumed* the aggregate remained a perfect gas when the surface opacity was changed, and he simply determined † the amount by which the luminosity would have to change in order to ensure the configuration remaining of diffuse type. As we have seen, change of surface opacity for given L and M has a disastrous effect on the structure of the diffuse configuration.

16. *Linear Series of Configurations.*—We can now trace the series of configurations of a given mass M of given opacity κ, on the standard

* There is one $L = L_0$ configuration which is stable for slight *decreases* of L, namely, that one whose central density is just large enough for degeneracy to be setting in there.

† I am unable to accept even this determination. It appears from Eddington's recent note (*M.N.*, **90**, 808, 1930) that his analysis never refers to the layer of changed opacity at all. To change the surface opacity and then omit to investigate the layer of changed opacity is scarcely to deal with the problem.

model, ϵ = constant, as the luminosity L varies from ∞ to zero. It will be seen that our results deduced mathematically are capable of the following interpretation. First of all let us restrict ourselves to configurations of perfect gas of unlimited compressibility. For $L > L_1$ no steady-state configuration exists; radiation-pressure exceeds gravity. For L just less than L_1 we must arrange our mass so as to give gravity the greatest possible effect. Suppose r_1 is the external radius. The matter between $r = r_1$ and $r = r_1 - \delta r$ will be subject to the maximum inward gravitational attraction when we distribute as much as possible of our mass M inside $r = r_1 - \delta r$, and as little as possible between $r = r_1$ and $r = r_1 - \delta r$. We therefore apportion a mass inside $(r_1, r_1 - \delta r)$ of vanishingly small density. We now do the same thing for the zone $(r_1 - \delta r, r_1 - 2\delta r)$, filling it with matter at as low a density as possible. We continue this process indefinitely. We finally arrive at a very small central sphere into which we have to pack nearly all the mass M. Thus the limiting perfect-gas configuration for $L \to L_1 - 0$ is an almost-point-mass M surrounded by a sphere of almost zero density of radius r_1. If we denote by $f_L(\xi)$ that solution of Emden's equation which corresponds to an assigned value of L, *i.e.* satisfying $(d\theta/d\xi_1) = -C^{-\frac{1}{2}}$, then as $L \to L_1 - 0$, $\theta = f_L(\xi)$ deforms continuously into the portion of the ξ-axis between 0 and 1 and the positive part of the θ-axis. As L decreases below L_1, the central condensation remains but becomes less pronounced. As $L \to L_0 + 0$, $\theta = f_L(\xi)$ deforms continuously into the Emden curve $\theta = f_{L_0}(\xi)$, together with the portion of the θ-axis lying above this curve. For $L = L_0$ the configuration changes discontinuously into the diffuse type. For $L < L_0$ radiation-pressure has become so low that no perfect-gas configurations are possible without artificial internal support. The opposite of a central condensation is a hole, and the only perfect-gas configurations are those consisting of a hollow shell supported internally by a spherical surface. The radius of the supporting surface is the radius of the first maximum of $\theta = f_L(\xi)$ as we proceed inwards from $\xi = 1$. As $L \to L_0 - 0$, $\theta = f_L(\xi)$ deforms continuously into the Emden curve $\theta = f_{L_0}(\xi)$, together with the portion of the θ-axis lying below this curve. Thus the hole begins as a vanishingly small one, at $\xi = 0$, and then increases in radius as L decreases below L_0.

All these configurations are of arbitrary external radius.

When we remove the restriction to perfect gas of unlimited compressibility, the configurations $L_1 > L > L_0$ will have a finite but large central density, and thus a degenerate core surrounded by a diffuse envelope; they prove to have a determinate radius * for each L. To the value $L = L_0$ will correspond, as before, diffuse configurations of arbitrary radius. To values $L_0 > L > 0$ will correspond "collapsed" configurations. By the circumstances of the case the latter will have a definite radius * for each L. It is clear that there is one configuration of the diffuse series $L = L_0$ which has the same central density as the limiting central density of collapsed configurations as $L \to L_0 - 0$. Thus the collapsed configurations and the diffuse configurations have a point

* This is verified later by calculation. See formulæ (61), (107).

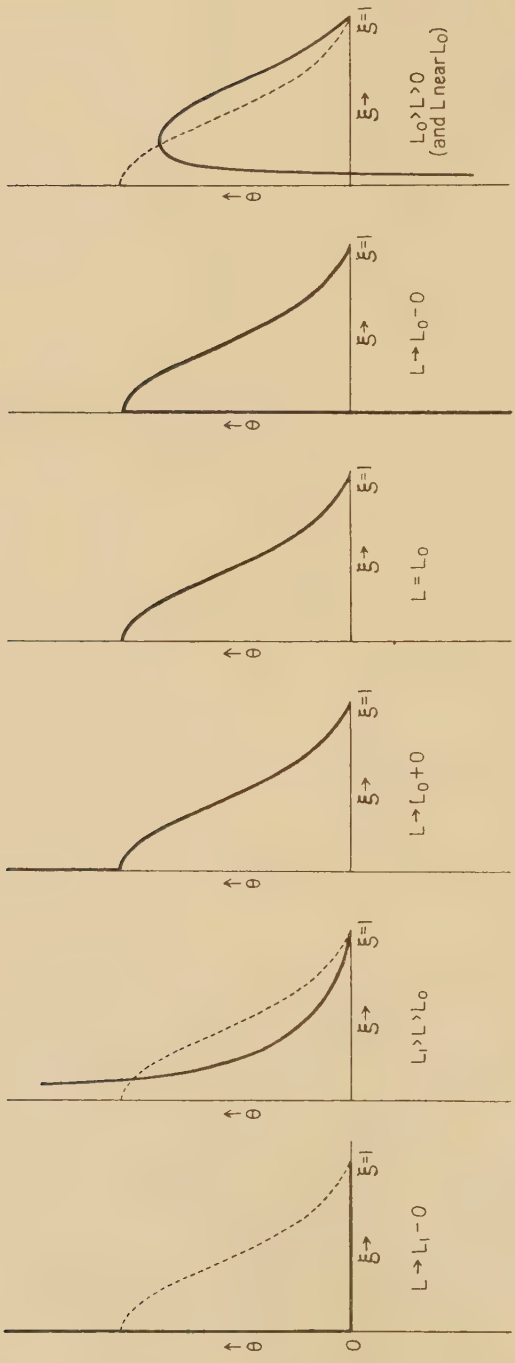

FIG. 2.—Forms of $\theta - f(\xi)$ for different values of L.

of bifurcation where the two linear series intersect and have a configuration in common (see fig. 4, p. 47).

Whether the linear series of centrally-condensed configurations $L_1 > L > L_0$ passes continuously to this same point of bifurcation as $L \to L_0 + 0$ is a matter of some importance. If passage through the central value $L = L_0$ from $L > L_0$ to $L < L_0$ is discontinuous, the gaseous envelope of the centrally-condensed configuration will collapse on to the central core as L passes through L_0; in doing so it will disengage a large quantity of gravitational potential energy in a very short time, and we shall have a phenomenon resembling a nova outburst. The investigation of this problem will be considered below. The passage through $L = L_0$ is certainly discontinuous for configurations of perfect gas of *unlimited* compressibility; for, for $L > L_0$ steady-state configurations of arbitrary radius exist, and for $L < L_0$ no steady-state configurations exist at all, the mass contracting indefinitely. Introduction of limited compressibility imposes a definite radius on the configurations $L_1 > L > L_0$ and $L_0 > L > 0$. The question is whether or not it introduces continuity of radius where before was discontinuity. If the passage remains discontinuous, we should have the suggestion that the phenomenon of a nova is the passage of a centrally-condensed star to a collapsed star, and that accordingly an ex-nova should be the early stage of a white dwarf. This could be settled by the observational determination of the density of an ex-nova or of the nucleus of a planetary nebula.

17. The discussion of the central regions of centrally-condensed and collapsed configurations can only be conducted by consideration of the next equation of state to be used. But the method is the same as the one already followed. We start with the new equation of state at the point where the classical gas-laws fail, and go inwards from this as far as we can. If we get to the centre, well and good. If not, we again change our equation of state as the physical conditions become more drastic. We can always finish up, in the last resort, with a homogeneous core at the maximum density of which matter is capable.

(C) *Standard Model, highly Collapsed* ($L << L_0$) *and completely Degenerate* (*Elementary Theory of White Dwarfs*).

18. *The Equation of State.*—All stars possess a gaseous fringe. But it is reasonable to begin our investigations of cored configurations with the case where L is so small, and collapse has consequently proceeded so far, that almost the whole of the system is in a state of non-perfect gas. (Later we shall show how to take account of the gaseous fringe.) We shall describe such a configuration as "highly collapsed." We continue to consider the standard model κ = constant, ϵ = constant. The configurations prove to have a high degree of homogeneity, so that the distribution ϵ = constant will probably correspond to nature.

We must now adopt some equation of state. From the results of many investigators, in real stars the state will be one of high ionization,

the material consisting of stripped nuclei and free electrons. It will be as nearly the "ideal gas" of theory as can occur in nature (apart from electrostatic corrections), but owing to its high density it will be an ideal gas in a "degenerate" state, as was first pointed out by R. H. Fowler * and used by Stoner,† Anderson,‡ and others. We use the term "degenerate" as referring to the Fermi-Dirac statistics,§ in the sense defined by Sommerfeld. Whether the compressed gas will be degenerate as regards both nuclei and electrons, or degenerate only as regards the electrons, is a matter to be investigated. Fortunately the nuclei in any case contribute little to the total pressure unless the material is principally protons and electrons; and even in the latter case the formulæ require only slight adjustments. For the sake of definiteness I shall consider a system of free nuclei, of atomic number Z and mass m, mixed with free electrons, of mass m_e, at the rate of Z electrons to every nucleus. The equation of state when the electrons alone are degenerate can be shown ‖ to be

$$p = \frac{\rho k\mathrm{T}}{m + \mathrm{Z}m_e} + \frac{1}{5}\left(\frac{3}{2}\right)^{\frac{2}{3}} \frac{h^2}{m_e(2\pi q_e)^{\frac{2}{3}}} \left(\frac{\mathrm{Z}\rho}{m + \mathrm{Z}m_e}\right)^{\frac{5}{3}}$$

where $q_e(=2)$ is the statistical weight of an electron. The first term is negligible. The second term is independent of T, and if as an approximation we put $m = 2\mathrm{Z}m_\mathrm{H}$ where m_H is the mass of the proton, we have approximately

$$p = \frac{1}{5}\left(\frac{3}{4\pi}\right)^{\frac{2}{3}} \frac{h^2}{(2m_\mathrm{H})^{\frac{5}{3}} m_e q_e^{\frac{2}{3}}} \rho^{\frac{5}{3}}.$$

This may be written

$$p = \mathrm{K}\rho^{\frac{5}{3}}. \quad . \quad . \quad . \quad . \quad (43)$$

where the numerical value of K is found to be

$$\mathrm{K} = 3{\cdot}17 \times 10^{12} \quad . \quad . \quad . \quad . \quad (44)$$

We note that K is independent of T and Z. This holds, of course, only for the limiting case of complete degeneracy. K may be called the degenerate-gas constant.

19. *Equations of the Problem.*—The equations of the problem for the standard model are (20), (21), and (22), for these do not depend upon the equation of state. Introducing (43) in (22′) we get

$$\mathrm{K}\rho^{\frac{5}{3}} = \frac{1}{3}a\mathrm{T}^4 \frac{\beta}{1-\beta} \quad . \quad . \quad . \quad . \quad (45)$$

* *M.N.*, **87**, 114, 1926, "Dense Matter."

† *Phil. Mag.*, **7**, 65, 1929; **9**, 944, 1930. Stoner's investigations of white dwarfs assumes that the only source of energy is gravitational contraction. As soon as the possibility of an internal supply of subatomic energy is admitted, his "maximum density" condition becomes invalid.

‡ *Zeits. für Phys.*, **66**, 851, 1929.

§ For full references, see my paper "The Dissociation-formula according to the Fermi-Dirac Statistics," *M.N.*, **90**, 769, 1930.

‖ *E.g.* by use of equation (39) in the paper last cited.

where β is defined by (23). The equation of mechanical equilibrium (20) becomes

$$\frac{K}{\beta}\frac{r^2}{\rho}\frac{d(\rho^{\frac{5}{3}})}{dr} = -GM(r) \quad . \quad . \quad . \quad (46)$$

or, differentiating,

$$\frac{5}{3}\frac{K}{G\beta}\frac{d}{dr}\left(r^2\rho^{-\frac{1}{3}}\frac{d\rho}{dr}\right) = -4\pi r^2\rho.$$

In this put

$$\rho = \lambda\psi^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where λ is to be chosen later. It becomes

$$\frac{5K}{8\pi G\beta}\lambda^{-\frac{1}{3}}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = -\psi^{\frac{3}{2}}.$$

In this put

$$r = \eta\left[\frac{5K}{8\pi G\beta}\lambda^{-\frac{1}{3}}\right]^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (48)$$

It becomes

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi}{d\eta}\right) = -\psi^{\frac{3}{2}}. \quad . \quad . \quad . \quad (49)$$

which is Emden's polytropic equation for "$n = \frac{3}{2}$."

20. *The Mass Relation.*—Substituting for ρ and r from (47) and (48) in (46) we find

$$M(r) = -\frac{1}{4(2\pi)^{\frac{1}{2}}}\left(\frac{5K}{G\beta}\right)^{\frac{3}{2}}\lambda^{\frac{1}{2}}\eta^2\frac{d\psi}{d\eta} \quad . \quad . \quad . \quad (50)$$

or substituting for β in terms of L and letting $M(r) \to M$, $r \to r_1$, $\eta \to \eta_1$,

$$\lambda = \frac{32\pi G^3M^2\left(1 - \frac{\kappa L}{4\pi cGM}\right)^3}{125K^3\eta_1{}^4(d\psi/d\eta)_1{}^2} \quad . \quad . \quad . \quad (51)$$

For a perfect gas the corresponding equation was (30), but there is a fundamental difference between (30) and (51), connected with the change of polytropic index from $n = 3$ to $n = \frac{3}{2}$. The symbol λ disappeared in forming (30), leaving $\xi_1{}^2(d\theta/d\xi)_1$, a determinate function of L and M. The symbol λ is retained in (51). The difference corresponds to the circumstance that for the perfect-gas case the radius is arbitrary, but, as we shall see, for the degenerate case the radius is definite. Equation (51) does *not*, as did (30), provide a starting value of the differential coefficient $d\psi/d\eta$.

We must now consider in turn the different solutions of Emden's polytropic equation for $n = \frac{3}{2}$, equation (49). We know simply that the discriminant C is small, since the configuration is highly collapsed.

21. *Case* (1).—Consider first Emden's solution, for which

$$\text{at} \quad \eta = 0, \qquad \psi = 1, \qquad d\psi/d\eta = 0 \quad . \quad (52)$$

Call this solution $\psi = \phi(\eta)$. If $\eta = \eta_0$ is the zero of $\phi(\eta)$, then by Emden's table (*Gaskugeln*, Tabelle 4, p. 79)

$$\eta_0 = 3{\cdot}6571, \qquad \phi'(\eta_0) = -0{\cdot}20316, \qquad \eta_0{}^2\phi'(\eta_0) = -2{\cdot}7176 \quad (53)$$

Now if $\psi = \Phi(\eta)$ is *any* solution of (49), so is $\psi = A^4\Phi(A\eta)$, where A is arbitrary. Hence $\psi = A^4\phi(A\eta)$ is a solution, and it vanishes when $\eta = \eta_1$, where $A\eta_1 = \eta_0$. For this solution the central density is given, from (47) and (51), by

$$\rho_c = \lambda[\psi^3]_{\eta=0} = \lambda A^6 = \frac{32\pi G^3M^2\left(1 - \frac{\kappa L}{4\pi cGM}\right)^3 A^6}{125K^3\eta_1{}^4[A^5\phi'(A\eta_1)]^2} \quad . \quad (54)$$

In this the factor involving η_1 and A reduces to

$$\frac{1}{A^4\eta_1{}^4[\phi'(A\eta_1)]^2}$$

$$= \frac{1}{\eta_0{}^4[\phi'(\eta_0)]^2},$$

Thus

$$\rho_c = \frac{32\pi G^3M^2\left(1 - \frac{\kappa L}{4\pi cGM}\right)^3}{125K^3\eta_0{}^4[\phi'(\eta_0)]^2} \quad . \quad . \quad . \quad (55)$$

where

$$\eta_0{}^4[\phi'(\eta_0)]^2 = (2{\cdot}7176)^2 = 7{\cdot}385.$$

We see that without loss of generality we may take A = 1, when λ reduces to ρ_c.

The mean density ρ_m is given by

$$\frac{\rho_m}{\rho_c} = \frac{4\pi\int_0^{\eta_0}\psi^3\eta^2 d\eta}{\frac{4}{3}\pi\eta_0{}^3} = -\frac{3}{\eta_0}\phi'(\eta_0) = \frac{1}{6{\cdot}0003} \quad . \quad (56)$$

Thus

$$\rho_m = \frac{96\pi G^3M^2\left(1 - \frac{\kappa L}{4\pi cGM}\right)^3}{125K^3\eta_0{}^5[-\phi'(\eta_0)]} \quad . \quad . \quad . \quad (57)$$

where

$$\eta_0{}^5[-\phi'(\eta_0)] = 132{\cdot}9.$$

As $L \to 0$ for constant M, ρ_c and ρ_m tend to limiting densities whose values are

$$\lim_{L\to 0}\rho_c = \frac{32\pi G^3M^2}{125K^3 \times 7{\cdot}385}, \qquad \lim_{L\to 0}\rho_m = \frac{96\pi G^3M^2}{125K^3 \times 132{\cdot}9}.$$

The central temperature T_c is given, from (45), by the equation

$$\frac{1}{3}aT_c{}^4\frac{\beta}{1-\beta} = K\rho_c{}^{\frac{4}{3}} \quad . \quad . \quad . \quad (58)$$

giving, from (55),

$$T_c = \frac{(4\pi)^{\frac{1}{3}}\left(\frac{\nu}{5}\right)^{\frac{2}{3}}GM^{\frac{1}{3}}}{K[-\eta_0{}^2\phi'(\eta_0)]^{\frac{2}{3}}}\left(\frac{\kappa L}{\frac{1}{3}ac}\right)^{\frac{1}{4}}\left(1 - \frac{\kappa L}{4\pi cGM}\right) \quad . \quad (59)$$

Thus $T_c \to 0$ as $L \to 0$.

The radius r_1 is given either by (48) or by $M = \frac{4}{3}\pi r_1^3 \rho_m$. We find in either case

$$r_1 = \eta_0 \left(\frac{5K}{8\pi G\beta}\right)^{\frac{1}{2}} \rho_c^{-\frac{1}{3}} \quad . \quad . \quad . \quad (60)$$

$$= \frac{5\eta_0[-\eta_0^2\phi'(\eta_0)]^{\frac{1}{3}}}{2^{\frac{2}{3}}\pi^{\frac{2}{3}}} \frac{K}{GM^{\frac{1}{3}}\left(1 - \frac{\kappa L}{4\pi cGM}\right)} \quad . \quad . \quad (61)$$

where $$\eta_0[-\eta_0^2\phi'(\eta_0)]^{\frac{1}{3}} = 5{\cdot}1032.$$

As $L \to 0$, r_1 steadily decreases to a definite limit. As M decreases, r_1 increases.

The effective temperature T_e is given by

$$(4\pi r_1^2)(\tfrac{1}{4}acT_e^4) = L$$

or

$$T_e = \frac{2^{\frac{5}{6}}\pi^{\frac{1}{2}}G^{\frac{1}{2}}M^{\frac{1}{3}}}{(5K)^{\frac{1}{2}}[-\eta_0^5\phi'(\eta_0)]^{\frac{1}{6}}} \left(\frac{L}{ac}\right)^{\frac{1}{4}} \left(1 - \frac{\kappa L}{4\pi cGM}\right)^{\frac{1}{2}} \quad . \quad (62)$$

where

$$[-\eta_0^5\phi'(\eta_0)]^{\frac{1}{3}} = 2{\cdot}259.$$

Thus $T_e \to 0$ as $L \to 0$. Also T_e decreases as M decreases.

22. *Application of Test for Degeneracy.*—One test remains to be applied. We have introduced the equation of state $p = K\rho^{\frac{5}{3}}$, but in order that this shall be valid it is necessary that Sommerfeld's criterion of degeneracy be satisfied. Now the classical equation of state being $p = (R/\mu)\rho T$, it is easily proved that the gas begins to be degenerate when $(R/\mu)\rho T = K\rho^{\frac{5}{3}}$, or $(R/\mu)T = K\rho^{\frac{2}{3}}$, or $\rho = (RT/K\mu)^{\frac{3}{2}}$. It is readily verified that in this notation Sommerfeld's condition for complete degeneracy is, in fact,

$$\rho\left(\frac{K\mu}{RT}\right)^{\frac{3}{2}} >> 1 \quad . \quad . \quad . \quad . \quad (63)$$

The star will have a perfect gas fringe, but this condition must be satisfied in the interior. Applying it to the centre, we find that a necessary condition for the validity of our analysis is

$$\frac{\rho_c}{T_c^{\frac{3}{2}}} >> \left(\frac{R}{K\mu}\right)^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad (64)$$

Hence, introducing (58) in (64),

$$\left[\frac{\frac{1}{3}a\rho_c}{K}\frac{\beta}{1-\beta}\right]^{\frac{3}{2}} >> \left(\frac{R}{\mu K}\right)^{\frac{3}{2}}$$

or

$$\left[\frac{\frac{1}{3}a\beta}{1-\beta}\frac{\rho_c K^3}{(R/\mu)^4}\right]^{\frac{3}{2}} >> 1 \quad . \quad . \quad . \quad (65)$$

But we know from (55) that

$$\rho_c = \frac{32\pi G^3 M^2 \beta^3}{125K^3[-\eta_0^2\phi'(\eta_0)]^2}.$$

Thus the condition (65) becomes

$$\left[\frac{\beta^4}{1-\beta}\left(\frac{R}{\mu}\right)^4 \cdot \frac{\frac{1}{3}\pi a G^3 M^2}{16} \cdot \frac{32 \times 16}{125 \times (2{\cdot}7176)^2}\right]^{\frac{3}{4}} >> 1$$

or, using the definition of the discriminant C given by (31),

$$\frac{1}{C^{\frac{3}{4}}} >> \left[\frac{125 \times (2{\cdot}7176)^2}{32 \times 16}\right]^{\frac{3}{4}}$$

or

$$\frac{1}{C} >> 1{\cdot}8 \quad . \quad . \quad . \quad . \quad (66)$$

But since the star by hypothesis is highly collapsed we must have by the analysis of section B the discriminant inequality

$$\frac{1}{C} >> \frac{1}{C_0} = \omega^2 = (2{\cdot}015)^2 \quad . \quad . \quad . \quad (67)$$

Thus (67) automatically provides for (66). In other words, the condition that the star is highly collapsed is identical with the condition that at the centre the material is completely degenerate. This is a very satisfactory result which confirms the general correctness of our line of thought. A wholly degenerate star is necessarily "collapsed."

23. *Case* (2).—Emden's equation (49) also possesses solutions which, near $\eta = 0$, have the asymptotic form

$$\psi \sim \frac{D}{\eta} . \quad . \quad . \quad . \quad . \quad (68)$$

where D is arbitrary.* It is necessary to enquire what configurations, if any, they represent.

If a complete configuration were represented by a solution with the behaviour (68) near $\eta = 0$, the configuration would necessarily require a point-mass at the origin, if D is positive. For by (50) and (68)

$$M(r) \propto -\eta^2 \frac{d\psi}{d\eta} \to +D \qquad \text{as } \eta \to 0.$$

Hence complete configurations represented by solutions with the asymptotic behaviour (68) are of no astrophysical interest.

We now consider whether a solution of type (68) can represent the *outer* part of a configuration of astrophysical interest. It is clear that as we pursue this solution inwards, either $M(r) \to$ a positive limit or $M(r)$ vanishes earlier. Thus these solutions, like perfect gas configurations, are either of centrally-condensed or of collapsed type, and we are driven to consider a new equation of state in the innermost region. The next simplest equation of state is that of homogeneous

* I owe this remark to Mr. R. H. Fowler. I had myself noticed simply that there is apparently a solution $\psi = A/\eta^4$, but, as Mr. Fowler showed me, the condition $A^{\frac{1}{2}} = -12$ fails to provide an actual solution. Solutions for which $\psi \sim D/\eta$ near $\eta = 0$ I propose to call "Fowler" solutions. The astrophysical interpretation which follows is my own.

incompressible material $\rho = \rho_{\max}$, where $\rho_{\max}$ is the maximum density of which matter is capable. The law $p = K\rho^{\frac{5}{3}}$ permits any density if the pressure is high enough, but we are bound to assume that the degree of compressibility is limited. Hence if the value ρ_c given by (55) exceeds a certain value $\rho_{\max}$, Emden's solution $\psi = \phi(\eta)$ cannot be continued to the centre, and we are therefore called upon to consider the fit of a degenerate envelope ($p = K\rho^{\frac{5}{3}}$) round an incompressible core ($\rho = \rho_{\max}$), the core and envelope being continuous in density at the interface. Let the interface be at $r = r'$, where $\eta = \eta'$ and $\psi = \psi'$.

The conditions are clearly, with our previous notation,

$$\rho_{\max} = \lambda\psi'^{\frac{5}{2}} \quad . \quad . \quad . \quad . \quad (69)$$

$$M(r') = \frac{4}{3}\pi r'^3\rho_{\max} = -\frac{1}{4(2\pi)^{\frac{1}{2}}}\left(\frac{5K}{G\beta}\right)^{\frac{3}{2}}\lambda^{\frac{1}{2}}\eta'^2\left(\frac{d\psi}{d\eta}\right)_{\eta=\eta'} \quad . \quad (70)$$

$$r' = \eta'\left(\frac{5K\lambda^{-\frac{1}{3}}}{8\pi G\beta}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (71)$$

by (47), (50), and (48). If from these we eliminate r' and λ we find all other physical constants disappear and we are left with

$$\frac{1}{3}\eta'^3\psi'^{\frac{3}{2}} = -\eta'^2\left(\frac{d\psi}{d\eta}\right)_{\eta=\eta'} \quad . \quad . \quad . \quad (72)$$

The problem now is to determine a solution $\psi \equiv \psi(\eta)$ such that (72) possesses a solution η'. The function $\psi(\eta)$ will then represent the configuration between the boundary $\eta = \eta_1$ and the interface $\eta = \eta'$; inside $\eta = \eta'$ we have $\rho = \rho_{\max}$.

We first show that ψ cannot be an Emden solution, *i.e.* cannot be of the type $\psi = A^4\phi(A\eta)$, where $\phi(\eta)$ (the function already considered in § 21) is such that $\phi(0) = 1$, $\phi'(0) = 0$. For from Emden's differential equation for "$n = \frac{3}{2}$," we have for *any* solution

$$-\eta^2\frac{d\psi}{d\eta} + \left[\eta^2\frac{d\psi}{d\eta}\right]_{\eta\to 0} = \int_0^\eta \psi^{\frac{3}{2}}\eta^2 d\eta \quad . \quad . \quad (73)$$

For Emden's solution, $\eta^2 d\psi/d\eta \to 0$ as $\eta \to 0$, and accordingly

$$-\eta^2\frac{d\psi}{d\eta} = \int_0^\eta \psi^{\frac{3}{2}}\eta^2 d\eta.$$

But if ψ is of Emden's type, $\psi(\eta) > \psi(\eta')$ in $0 < \eta < \eta'$. Hence

$$-\eta'^2\left(\frac{d\psi}{d\eta}\right)_{\eta=\eta'} > \left[\psi(\eta')\right]^{\frac{3}{2}}\int_0^{\eta'}\eta^2 d\eta = \frac{1}{3}\eta'^3\psi'^{\frac{3}{2}},$$

which proves that, if ψ is of Emden's type, (72) has no root η' save $\eta' = 0$. In the latter case $\rho_c = \rho_{\max}$ and the configuration is completely represented by a ψ of Emden's type. It is the densest of the degenerate series devoid of homogeneous core.

Next consider solutions of "$n = \frac{3}{2}$" of Fowler type. In these, either (1), $\psi(\eta) > 0$ in $0 < \eta < \eta_1$, and near $\eta = 0$, $\psi(\eta) \sim D/\eta$, where

$D > 0$; or (2), $\psi(\eta)$ has a zero in $0 < \eta < \eta_1$. We first show that the configuration cannot be represented by the first alternative. For if it were, $\eta^2 d\psi/d\eta \to -D$ as $\eta \to 0$, where $D > 0$, and accordingly for any η, by (73),

$$-\eta^2\frac{d\psi}{d\eta} - D = \int_0^\eta \psi^{\frac{3}{2}}\eta^2 d\eta.$$

This implies

$$-\eta^2\frac{d\psi}{d\eta} - D > \frac{1}{3}\eta^3\psi^{\frac{3}{2}},$$

which shows that if (72) has a root η' then $-D > 0$ or $D < 0$. This contradicts the hypothesis $D > 0$. Hence ψ must be of the type (2),

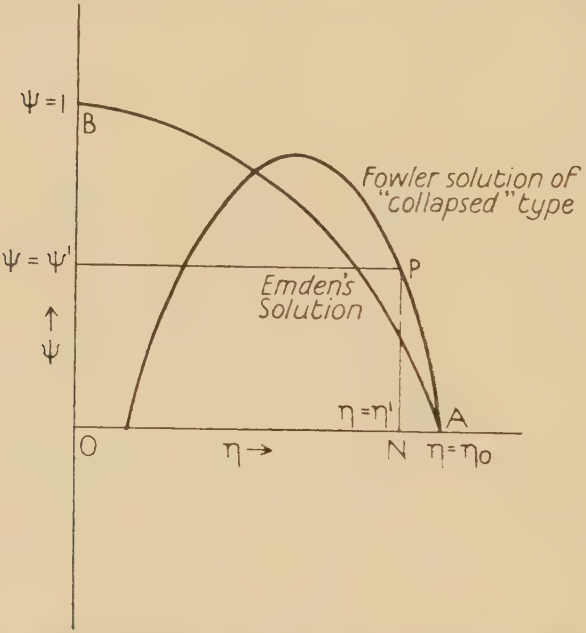

Fig. 3.—Graphs of solutions of $\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi}{d\eta}\right) = -\psi^{\frac{3}{2}}$ through $\eta = \eta_0$, $\psi = 0$.

which has a zero in $0 < \eta < \eta_1$, *i.e.* is of collapsed type. The state of affairs is shown in fig. 3.

We can now show that to each Fowler solution of "collapsed type" (type (2)) there corresponds a configuration consisting of a degenerate envelope surrounding a homogeneous core.

Let $\psi = \phi(\eta)$ be a *particular* Fowler solution of this type. Consider the associated family of solutions $\psi = A^4\phi(A\eta)$, where A is arbitrary. Let η_0 be the zero of $\phi(\eta)$; then the function $\psi = A^4\phi(A\eta)$ vanishes at $\eta = \eta_1$, where $A\eta_1 = \eta_0$. Inserting in (72), η' must be a root of

$$\tfrac{1}{3}\eta'^3[A^4\phi(A\eta')]^{\frac{3}{2}} = -\eta'^2 A^5\phi'(A\eta')$$

or, putting $A\eta' = \sigma$,

$$\tfrac{1}{3}\sigma[\phi(\sigma)]^{\frac{1}{3}} = -\phi'(\sigma) \quad . \quad . \quad . \quad . \quad (74)$$

This is an equation for the pure number σ. Mr. Fowler has kindly sent me a manuscript proof that for Fowler solutions of "$n = \frac{3}{2}$" of collapsed type this equation always has a root σ; σ, of course, depends upon the particular Fowler solution chosen, *e.g.* on its slope at A (fig. 3).

Now (51) depends only on the nature of the envelope as we approach the outer boundary, and so gives the value of λ in the case of an envelope surrounding a homogeneous core just as well as it did in the absence of a homogeneous core. Inserting this value of λ in (69) we get

$$\rho_{\max} = \frac{32\pi G^3M^2\left(1 - \dfrac{\kappa L}{4\pi cGM}\right)^3}{125K^3} \cdot \frac{[A^4\phi(A\eta')]^{\frac{3}{2}}}{\eta_1{}^4[A^5\phi'(A\eta_1)]^2}$$

$$= \frac{32\pi G^3M^2\left(1 - \dfrac{\kappa L}{4\pi cGM}\right)^3}{125K^3} \cdot \frac{[\phi(\sigma)]^{\frac{3}{2}}}{\eta_0{}^4[\phi'(\eta_0)]^2} \quad . \quad . \quad (75)$$

In this $\rho_{\max}$ is a definite number, namely, the assumed maximum density of matter. Suppose now κ and L are given. Then for a given Fowler solution $\psi = \phi(\eta)$ of the type (2) this is a definite equation for M, with one real root. Hence to each ϕ for given κ and L there corresponds a definite total mass M. And the mass and radius of the core and the radius of the star, for this ϕ, κ, L, M, can readily be written down from formulæ already given. Thus to any Fowler solution $\phi(\eta)$ of collapsed type there corresponds a determinate configuration consisting of a degenerate envelope surrounding a homogeneous core of density $\rho_{\max}$, the density of the envelope being continuous with that of the core at the interface.

The converse problem of finding the configuration for given κ, L, M is equivalent to that of finding a Fowler solution $\psi = \phi(\eta)$ of type (2), for which

$$\frac{\eta_0{}^4[\phi'(\eta_0)]^2}{[\phi(\sigma)]^{\frac{3}{2}}} = \frac{32\pi G^3M^2\left(1 - \dfrac{\kappa L}{4\pi cGM}\right)^3}{125K^3\rho_{\max}} \quad . \quad . \quad (76)$$

where η_0 is the "boundary" zero of $\phi(\eta)$ and σ is the root of the equation

$$\tfrac{1}{3}\sigma[\phi(\sigma)]^{\frac{3}{2}} = -\phi'(\sigma),$$

it being given that the value ρ_c computed from (55) exceeds $\rho_{\max}$ when the $\phi(\eta)$ in (55) is Emden's solution. Mr. R. H. Fowler has kindly studied this problem at my request, and has constructed a rigorous proof that the function $\phi(\eta)$ can always be found to suit these conditions.

It is clear that for given κ, L, as M increases the degenerate Emden configurations (without homogeneous core) pass continuously into

configurations consisting of a degenerate envelope (described by a Fowler solution) surrounding a homogeneous core, the mean density steadily increasing. For given κ and M, as L decreases the central density of the degenerate Emden configurations steadily increases, but it may or may not reach $\rho_{\max}$ according to the value of M.

The application of this analysis will be seen when we come to consider composite configurations consisting of a gaseous envelope surrounding a degenerate core. The degenerate core may then be itself composite, and include an inner homogeneous core of density $\rho = \rho_{\max}$.

24. *Numerical Application to White Dwarfs.**—We have now solved our problem of finding the steady-state configuration of a highly collapsed, completely degenerate gas of arbitrary M, L, κ, on the standard model. In accordance with our general line of argument, our analysis provides formulæ for ρ_c, ρ_m, T_c, r_1, T_e in terms simply of the variables κ, L, M and physical constants. The formulæ are valid, subject to the conditions that $4\pi cGM > \kappa L$ and that the discriminant C is sufficiently small. We observe from (55) that the central density ρ_c increases as M increases, and increases as L decreases. The mean density is as much as $\frac{1}{6}$ of the central density, so that the aggregate is fairly homogeneous. The central temperature T_c by (59) depends not only on L through the factor $\beta(= 1 - \kappa L/4\pi cGM)$, but also on L through the factor $L^{\frac{1}{4}}$. Thus as L decreases, T_c decreases indefinitely. Thus as L decreases, degeneracy becomes more pronounced. The radius r_1, by (61), decreases as L decreases, but increases as M decreases. The effective temperature T_e by (62) decreases as L decreases, but only slowly, namely, as $L^{\frac{1}{4}}$. We see that our formulæ are in accordance with the common-sense view that the lower the luminosity the colder the star, both at its surface and in its interior. Further, T_c increases with κ; the more opaque the material the hotter it is. The ratio of central temperature to effective temperature is, by (59) and (62),

$$\frac{T_c}{T_e} = \frac{2^{\frac{5}{2}}\pi^{\frac{1}{2}}G^{\frac{1}{2}}M^{\frac{5}{12}}\kappa^{\frac{1}{4}}\left(1 - \dfrac{\kappa L}{4\pi cGM}\right)^{\frac{1}{4}}}{(\frac{1}{35})^{\frac{1}{4}}5^{\frac{3}{4}}K^{\frac{1}{4}}[\eta_0{}^5(\phi'(\eta_0))^4]^{\frac{1}{4}}} \quad . \quad . \quad (77)$$

Thus when $\kappa L/4\pi cGM$ is small, T_c/T_e increases with M and with κ.

We now apply our formulæ to a star equal in mass to $0{\cdot}85\odot$, that of the Companion of Sirius. We cannot evaluate at present $\kappa L/4\pi cGM$, as the problem of the theoretical opacity κ of a degenerate gas has not yet been attempted; we certainly cannot use the perfect-gas formula $\kappa \propto \rho/T^{\frac{7}{2}}$. We therefore leave $\kappa L/4\pi cGM$ un-evaluated. It will be noticed that all the formulæ can be worked out numerically (apart from the β factors) in terms of M alone until we come to T_e, which requires a knowledge of L. We take this to be $\frac{1}{3\,6\,0}\odot$.† It is simplest not to use the explicit formulæ, save for ρ_c, but to calculate

* This section refers to configurations without homogeneous core, *i.e.* to §§ 21, 22

† Data for the Companion of Sirius are taken from Eddington's *Internal Constitution of the Stars*, p. 171.

in turn ρ_m, T_c from (58), r_1 from $\frac{4}{3}\pi r_1^3\rho_m = M$, and T_e from $\pi a c r_1^2 T_e^4 = L$. The results are as follows :—

$$\begin{aligned}
\rho_c &= 2{\cdot}87 \times 10^6\beta^3 \text{ gram cm.}^{-3},\\
\rho_m &= 4{\cdot}79 \times 10^5\beta^3 \text{ gram cm.}^{-3},\\
T_c &= 2{\cdot}92 \times 10^9\beta(1-\beta)^{\frac{1}{4}} \text{ degrees},\\
r_1 &= 9{\cdot}44 \times 10^8\beta^{-1} \text{ cms.},\\
T_e &= 11{,}300\beta^{\frac{1}{2}} \text{ degrees},
\end{aligned}$$

where it must be remembered that $\beta = 1 - \kappa L/4\pi cGM$, and so is less than unity. The "observed" mean density of the Companion of Sirius is $6{\cdot}1 \times 10^4$, the "observed" radius, $1{\cdot}88 \times 10^9$ cms., the "observed" effective temperature, 8,000° – 10,000°. The Companion of Sirius cannot be completely degenerate, it must have a gaseous fringe, and it will not necessarily be built on the standard model κ = constant, ϵ = constant. But a mere glance at our results shows that the highly collapsed, completely degenerate standard model closely resembles the Companion of Sirius, and this is sufficient for us to identify the "collapsed" configurations of our general theory with the white dwarfs of nature.* The gaseous fringe will have the effect of decreasing the mean density, increasing the radius, and decreasing the effective temperature as compared with our theoretical calculations—in all cases lifting the figures in the direction of those observed for the Companion of Sirius. Once we have made this identification, we can infer from the fair degree of homogeneity of our model in density and from the temperature law $T^4 \propto \rho^3$ or $T \propto \rho^{\frac{3}{4}}$ that the physical conditions must be roughly uniform throughout a white dwarf, so that ϵ may be expected to be fairly uniform. This justifies our close study of the standard model as far as ϵ is concerned; whether κ is fairly constant or not can only be discovered by study of the absorption coefficient of a degenerate gas.

It is to be particularly noticed that we give no formula for L. L is one of the data of the problem, not a quantity that can be in any way deduced from steady-state considerations without knowledge of the energy-generating process. Having, however, ascertained the physical conditions inside a white dwarf we are now in a position to discuss the physics of energy-generation. If this can be made to yield the rate of energy-generation, L will be determinate and the luminosity known. I shall not attempt in this paper to discuss the physics of energy-generation. I merely emphasise that close discussion of the highly collapsed, completely degenerate standard model fully justifies all the conclusions obtained on more general grounds in the introduction.

Instead of computing ρ_m, etc., from the theoretical value of the

* It is probable that the effective temperature T_e of any white dwarf is much higher than would be inferred from its spectral type; for the surface gravity must be very high, the reversing layer pressure high, the ionization low, and so the type later than would be computed from T_e. It would be highly interesting to determine observationally the colour-indices of the white dwarfs and see if they are abnormally low for the type. The effect of a higher value for T_e is to give greater surface brightness, and so, for the observed L, a smaller computed radius and a higher computed mean density, than ordinarily given.

degenerate-gas constant K, we could use the observed ρ_m for a white dwarf to deduce an "astronomical" value of K. Since the observable which actually discloses the white-dwarf character of a star is the effective temperature T_e, and since by (62) the deduced value of K depends upon T_e [2], we have a fairly accurate determination of K, assuming that L and M can be observed accurately and that $\kappa L/4\pi cGM$ is small. The whole investigation thus affords a valuable check on the quantum theory of the degenerate gas.

(D) *Standard Model, Configurations consisting of Perfect-gas Envelope surrounding a Degenerate Core. (Refined theory of white dwarfs. First discussion of theory of giants and ordinary dwarfs.)*

25. We proceed to investigate the steady-state configurations of the standard model with a gaseous outer envelope and a degenerate core. The collapsed configurations of this type may be expected to correspond very closely to the actual white dwarfs of nature. The centrally-condensed configurations will have to be compared with the giants and ordinary dwarfs of nature. For brevity we shall call configurations consisting of a perfect-gas envelope surrounding a degenerate core, "composite configurations." We now give the general theory of composite configurations built on the standard model. It is probable, of course, that if centrally-condensed composite configurations correspond to giants and ordinary dwarfs, then the energy-sources will be located in the central condensation and not be uniformly distributed (for the collapsed configurations the standard model will be appropriate). But owing to the analytical difficulties of point-source models we study first the standard model for composite configurations of both centrally-condensed and collapsed types.

26. *Surface of Separation between Perfect-gas Envelope and Degenerate Core.*—Though the actual transition from the perfect-gas envelope to the degenerate central region must occupy a certain zone, we may for convenience take the two regions to be separated by a definite surface of demarcation. Outside this surface, $p = (R/\mu)\rho T$; inside it, $p = K\rho^{\frac{5}{3}}$ (if necessary, inside this again we may have a core $\rho = \rho_{max}$). At the surface of separation p must be continuous, and so here

$$p = (R/\mu)\rho T = K\rho^{\frac{5}{3}} \quad . \quad . \quad . \quad (78)$$

Let ρ', T' be the values of ρ and T at this interface. The relation

$$p = \tfrac{1}{3}aT^4\frac{\beta}{1-\beta} \quad . \quad . \quad . \quad . \quad (79)$$

holds for the standard model whatever the equation of state. Solving (78) and (79) we find at the interface

$$\rho' = \frac{(R/\mu)^4}{\frac{1}{3}aK^3}\frac{1-\beta}{\beta} \quad . \quad . \quad . \quad . \quad (80)$$

$$T' = \frac{(R/\mu)^{\frac{5}{3}}}{(\frac{1}{3}a)^{\frac{1}{3}}K}\left(\frac{1-\beta}{\beta}\right)^{\frac{2}{3}} \quad . \quad . \quad . \quad (81)$$

These can be evaluated directly, apart from the factor involving β. Taking a molecular weight $\mu = 2m_H$, with the values

$$R/\mu = 1{\cdot}37 \times 10^{-16}/2 \times 1{\cdot}65 \times 10^{-24},$$
$$K = 3{\cdot}17 \times 10^{12}, \qquad \tfrac{1}{3}a = 2{\cdot}55 \times 10^{-15},$$

we find

$$\rho' = 3{\cdot}66 \times 10^{7}\frac{1-\beta}{\beta} \text{ gram cm.}^{-3} \quad . \quad . \quad (82)$$

$$T' = 0{\cdot}843 \times 10^{10}\left(\frac{1-\beta}{\beta}\right)^{\frac{1}{3}} \text{ degrees} \quad . \quad . \quad (83)$$

where $(1-\beta)/\beta = \kappa L/(4\pi cGM - \kappa L)$. The factor $(1-\beta)/\beta$ may have any value. It will be large for "highly centrally condensed" types, small for "highly collapsed" types of configuration. But the order of magnitude of ρ' and T′ is at least disclosed. The densities and temperatures in the core must be higher than ρ' and T′. We thus see that the greater L is for given κ and M, and accordingly the more centrally condensed, the greater are ρ' and T′. Thus the cores of very luminous configurations are denser and hotter than those of less luminous configurations. This is a major result of our analysis. Again, ρ' and T′ increase with κ.

27. *The Equations of Fit.*—Let $r = r'$ be the radius of the surface of demarcation. Outside $r = r'$ the distribution of density will be given by a solution of Emden's equation for "$n = 3$"; inside $r = r'$ it will be given by a solution of Emden's equation for "$n = \frac{3}{2}$." This follows because in each case the reduction to an Emden equation depends only on the validity of equation (79), which holds whatever the equation of state.

The fit of the "$n = 3$" distribution on to the "$n = \frac{3}{2}$" distribution must be made in such a way that the density of each distribution at the interface is ρ', and the mass missing from the incomplete "$n = 3$" distribution must be actually found in the incomplete "$n = \frac{3}{2}$" distribution.

We shall adopt for the "$n = 3$" region a notation similar to that used in discussing the complete "$n = 3$" distribution in Section B, with the one difference that we shall replace λ by the symbol λ_1. We shall adopt for the "$n = \frac{3}{2}$" region a notation similar to that used in discussing the complete "$n = \frac{3}{2}$" distribution in Section C, with the one exception that we shall replace λ by λ_2. Relation (29) holds in the "$n = 3$" region and we have accordingly

$$M = -\frac{4}{(\frac{1}{3}\pi aG^3)^{\frac{1}{2}}}\left(\frac{R}{\mu}\right)^2\frac{(1-\beta)^{\frac{1}{2}}}{\beta^2}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \quad . \quad . \quad (84)$$

$$M(r') = -\frac{4}{(\frac{1}{3}\pi aG^3)^{\frac{1}{2}}}\left(\frac{R}{\mu}\right)^2\frac{(1-\beta)^{\frac{1}{2}}}{\beta^2}\xi'^2\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi'} \quad . \quad . \quad (85)$$

where $\xi = \xi'$ is the interface given, by (27), by

$$r' = \frac{\xi'}{\lambda_1}\frac{R}{\mu}\frac{(1-\beta)^{\frac{1}{2}}}{\beta}\frac{1}{(\frac{1}{3}\pi aG)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (86)$$

Relation (50) holds in the "$n = \frac{3}{2}$" region and we have accordingly

$$M(r') = -\frac{1}{4(2\pi)^{\frac{1}{2}}}\left(\frac{5K}{G\beta}\right)^{\frac{3}{2}}\lambda_2^{\frac{1}{2}}\eta'^2\left(\frac{d\psi}{d\eta}\right)_{\eta=\eta'} \quad . \quad . \quad (87)$$

where by (48) $\eta = \eta'$ is the same interface, given by

$$r' = \eta'\left(\frac{5K}{8\pi G\beta}\right)^{\frac{1}{2}}\lambda_2^{-\frac{1}{6}} . \quad . \quad . \quad . \quad (88)$$

From the distribution in the "$n = 3$" region the density there is given by (24) and (27) in the form

$$\rho = \frac{\frac{1}{3}a}{R/\mu}\frac{\beta}{1-\beta}\lambda_1^3\theta^3 . \quad . \quad . \quad . \quad (89)$$

and from the distribution in the "$n = \frac{3}{2}$" region the density there is given, by (47), by

$$\rho = \lambda_2\psi^{\frac{3}{2}} . \quad . \quad . \quad . \quad . \quad (90)$$

These two distributions must both take the value ρ' at the interface, as given by (80). Denoting $\theta(\xi')$ by θ' and $\psi(\eta')$ by ψ' we have accordingly

$$\frac{(R/\mu)^4}{\frac{1}{3}aK^3}\frac{1-\beta}{\beta} = \frac{\frac{1}{3}a}{R/\mu}\frac{\beta}{1-\beta}\lambda_1^3\theta'^3 . \quad . \quad . \quad (91)$$

and

$$\frac{(R/\mu)^4}{\frac{1}{3}aK^3}\frac{1-\beta}{\beta} = \lambda_2\psi'^{\frac{3}{2}} \quad . \quad . \quad . \quad (92)$$

The functions $\theta(\xi)$ and $\psi(\eta)$ satisfy certain differential equations, namely, the Emden equations for "$n = 3$" and "$n = \frac{3}{2}$" respectively, and we must consider what solutions to choose. Let $f(\xi)$ be a solution of

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3 . \quad . \quad . \quad . \quad (93)$$

vanishing at $\xi = \xi_0$ and satisfying the condition (84), namely,

$$\xi_0^2 f'(\xi_0) = -\frac{(\frac{1}{3}\pi aG^3)^{\frac{1}{2}}M}{4}\left(\frac{\mu}{R}\right)^2\frac{\beta^2}{(1-\beta)^{\frac{1}{2}}} \quad . \quad . \quad (94)$$

Then by the usual argument $\theta = Af(A\xi)$ is a solution of (93) vanishing at $\xi = \xi_1$, where $A\xi_1 = \xi_0$, and, as is easily verified, satisfying (84), since $\xi_1^2(d\theta/d\xi)_1 = A^2\xi_1^2 f'(A\xi_1) = \xi_0^2 f'(\xi_0)$. Write $A\xi' = a_1$. Then (85) gives

$$M(r') = -\frac{4}{(\frac{1}{3}\pi aG^3)^{\frac{1}{2}}}\left(\frac{R}{\mu}\right)^2\frac{(1-\beta)^{\frac{1}{2}}}{\beta^2}a_1^2 f'(a_1) \quad . \quad (95)$$

(86) gives

$$r' = \frac{a_1}{A\lambda_1}\frac{R}{\mu}\frac{(1-\beta)^{\frac{1}{2}}}{\beta}\frac{1}{(\frac{1}{3}\pi aG)^{\frac{1}{2}}} \quad . \quad . \quad (96)$$

and (91) gives

$$\frac{(R/\mu)^5}{(\frac{1}{3}a)^2K^3}\frac{(1-\beta)^2}{\beta^2} = (A\lambda_1)^3[f(a_1)]^3 \quad . \quad . \quad (97)$$

Eliminating $A\lambda_1$ between (96) and (97) we get

$$r'^3 = \frac{(\frac{1}{3}a)^{\frac{1}{2}}}{(\pi G)^{\frac{3}{2}}} \frac{K^3}{(R/\mu)^2} \frac{1}{\beta(1-\beta)^{\frac{1}{2}}} a_1^3[f(a_1)]^3 \quad . \quad . \quad (98)$$

The function $f(\xi)$, we must remember, depends upon the discriminant C according to (94), and therefore upon L, M, and κ.

Next let $\phi(\eta)$ be *any* solution of

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi}{d\eta}\right) = -\psi^5 \quad . \quad . \quad . \quad (93')$$

and let it vanish at $\eta = \eta_0$.

Then $\psi = B^4\phi(B\eta)$ is also a solution. It vanishes at $\eta = \eta_1$, where $B\eta_1 = \eta_0$. Write $B\eta' = b_1$. Then (92) gives

$$\frac{(R/\mu)^4}{\frac{1}{3}aK^3}\frac{1-\beta}{\beta} = \lambda_2 B^6[\phi/b_1)]^5 \quad . \quad . \quad . \quad (99)$$

Then by (87)

$$M(r') = -\frac{1}{4(2\pi)^{\frac{1}{2}}}\left(\frac{5K}{G\beta}\right)^{\frac{3}{2}}\lambda_2^{\frac{1}{2}}\left(\frac{b_1}{B}\right)^2 B^5\phi'(b_1)$$

or, eliminating $\lambda_2^{\frac{1}{2}}B^3$ by (99),

$$M(r') = -\frac{5^{\frac{3}{2}}}{2^{\frac{7}{2}}}\frac{(R/\mu)^2}{(\frac{1}{3}\pi aG^3)^{\frac{1}{2}}}\frac{(1-\beta)^{\frac{1}{2}}}{\beta^2}\frac{b_1^2\phi'(b_1)}{[\phi(b_1)]^{\frac{5}{2}}} \quad . \quad . \quad (95')$$

And by (88)

$$r' = \frac{b_1}{B}\left(\frac{5K}{8\pi G\beta}\right)^{\frac{1}{2}}\lambda_2^{-\frac{1}{6}}$$

or, eliminating $\lambda_2^{\frac{1}{6}}B$ by means of (99),

$$r'^3 = \frac{5^{\frac{3}{2}}}{8^{\frac{3}{2}}}\frac{(\frac{1}{3}a)^{\frac{1}{2}}}{(\pi G)^{\frac{3}{2}}}\frac{K^3}{(R/\mu)^2}\frac{1}{\beta(1-\beta)^{\frac{1}{2}}}b_1^3[\phi(b_1)]^{\frac{1}{2}} \quad . \quad . \quad (98')$$

The conditions of fit are that the values of $M(r')$ given by (95) and (95′) shall be equal, and that the values of r' given by (98) and (98′) shall be equal. Equating the values, we see that all physical constants disappear and we find ourselves left simply with the equations:

$$4a_1^2f'(a_1) = \frac{5^{\frac{3}{2}}}{2^{\frac{7}{2}}}\frac{b_1^2\phi'(b_1)}{[\phi(b_1)]^{\frac{5}{2}}},$$

$$a_1^3[f(a_1)]^3 = \frac{5^{\frac{3}{2}}}{8^{\frac{3}{2}}}b_1^3[\phi(b_1)]^{\frac{1}{2}}$$

which may be written more concisely in the forms:

$$\left(\frac{8}{5}\right)^{\frac{3}{2}}a_1^2f'(a_1) = \frac{b_1^2\phi'(b_1)}{[\phi(b_1)]^{\frac{5}{2}}} \quad . \quad . \quad . \quad . \quad (100)$$

$$\left(\frac{8}{5}\right)^{\frac{1}{2}}a_1f(a_1) = b_1[\phi(b_1)]^{\frac{1}{2}} \quad . \quad . \quad . \quad (101)$$

(These equations possess a property of invariance. If we write

$$a_1 = \gamma A_1 \quad \gamma f(\gamma A_1) = F(A_1) \quad \gamma^2 f'(\gamma A_1) = F'(A_1),$$
$$b_1 = \delta B_1 \quad \delta^4 \phi(\delta B_1) = \Phi(B_1) \quad \delta^5 \phi'(\delta B_1) = \Phi'(B_1),$$

the equations reduce to

$$\left(\frac{8}{5}\right)^{\frac{3}{2}} A_1^2 F'(A_1) = \frac{B_1^2 \Phi'(B_1)}{[\Phi(B_1)]^{\frac{3}{4}}},$$
$$\left(\frac{8}{5}\right)^{\frac{1}{2}} A_1 F(A_1) = B_1 [\Phi(B_1)]^{\frac{1}{4}},$$

which are of the same form whatever the constants γ and δ. But F and Φ are also solutions of "$n = 3$" and "$n = \frac{3}{2}$" respectively. It will be seen that we arrive at the same values of all physical quantities ρ_c, r_1, etc., whatever member f or ϕ of a given "family" of solutions we choose; all our physical formulæ will be found to possess the same invariance property.)

28. In (100) and (101) we have a pair of simultaneous equations for the two unknowns (pure numbers) a_1 and b_1. The function $f(\xi)$ is definite. It is determined as that solution of Emden's "$n = 3$" equation through $\xi = \xi_0$, $\theta = 0$, which satisfies the discriminant equation (94). It will be of centrally-condensed type, Emden type, or collapsed type, according as $L_1 > L > L_0$, $L = L_0$, or $L_0 > L > 0$. The function $\phi(\eta)$ is not yet definite. It is any solution of Emden's "$n = \frac{3}{2}$" equation through $\eta = \eta_0$, $\theta = 0$. It may be Emden's solution itself, of the type $\phi \to$ constant, $d\phi/d\eta \to 0$ as $\eta \to 0$, or it may be a Fowler solution of collapsed type, with a zero in $0 < \eta < \eta_0$. In the latter case, in accordance with the investigations of Section C, paragraph 23, the equation of state must change again or the configuration could only exist with an artificial internal supporting surface, and the simplest change of equation of state we can introduce is to consider homogeneous material of density $\rho = \rho_{\max}$, $\rho_{\max}$ being supposed known from the physical properties of matter.

We have therefore to consider whether (100) and (101) have solutions according to the different forms of f and the different forms of ϕ. This is a question requiring a paper to itself, and I have not completed my investigations.* I propose to consider here certain cases only, leaving the general discussion to a further occasion.

If f is Emden's solution of "$n = 3$" ($L = L_0$) and ϕ is Emden's solution of "$n = \frac{3}{2}$," the only solution of (100) and (101) appears to be $a_1 = 0$, $b_1 = 0$, and the configuration is an Emden-Eddington polytrope with central density $\rho_c = \rho'$. The degenerate core reduces to a point.

If f is a solution of "$n = 3$" of collapsed type and ϕ is Emden's solution of "$n = \frac{3}{2}$," examination of the equations shows that roots a_1 and b_1 exist, and that $a_1 \to 0$, $b_1 \to 0$ as $L \to L_0 - 0$. Further, as $L \to 0$, the degenerate core spreads until it occupies the whole con-

* A complete set of tables of the functions f and φ is required, of the different types. A set of tables of f is given in Mr. Fairclough's paper in this number of *M.N.* Tables of φ await construction.

figuration, and we have $a_1 \to \xi_0$, $b_1 \to \eta_0$. This gives the series of composite collapsed configurations corresponding to white dwarfs. The case is worked out in detail below, § 30, Case (1).

If f is of collapsed type and the central density ρ_c as calculated for Emden's ϕ exceeds $\rho_{\max}$, then we are compelled to consider a ϕ of Fowler type; it yields a core of density $\rho_{\max}$ inside a degenerate envelope, the whole inside a gaseous envelope. This case is also considered in detail below, § 30, Case (2). These configurations would correspond to massive stars in the dense state.

If f is a solution of "$n = 3$" of centrally-condensed type and ϕ is Emden's solution of "$n = \frac{3}{2}$," I have not been able to convince myself either analytically or by drawing graphs that (100) and (101) possess solutions in a_1 and b_1. If this is confirmed, the physical reason is obvious. In the centrally-condensed gaseous envelope the density near the centre is rising so rapidly that the necessary mass cannot be contained in a degenerate Emden core of Emden type "$n = \frac{3}{2}$." We are then driven to consider a Fowler solution of "$n = \frac{3}{2}$" of collapsed type, which gives a more rapid initial rise of density in the degenerate core; inside this degenerate zone, to avoid collapse, we must then insert a core of a new equation of state, say $\rho = \rho_{\max}$. Formally, this is also represented by the analysis of § 30, Case (2) below. I anticipate that these configurations, when fully studied, will be found to correspond to giants and ordinary dwarfs; it is clear that the solution a_1 must be very small, $f(a_1)$ very large, and so, as the formulæ will indicate, the radius r_1 very large and ρ_m very small, compared with the value for collapsed configurations.

29. *Physical Properties of the Configurations.*—In all these cases, whenever a_1 and b_1 can be determined, the configuration becomes definite, and the physical quantities ρ_c, etc., can be written down. The following formulæ are all valid when ϕ is Emden's solution of "$n = \frac{3}{2}$." All save those giving ρ_c and T_c are also valid if ϕ is a Fowler solution of "$n = \frac{3}{2}$" of collapsed type.

We have $\rho_c = \lambda_2 B^6$, since $\phi(0) = 1$ when ϕ is Emden's solution of "$n = \frac{3}{2}$." Hence by (99)

$$\rho_c = \frac{(R/\mu)^4}{\frac{1}{3}aK^3} \frac{1-\beta}{\beta} \frac{1}{[\phi(b_1)]^3} \quad . \quad . \quad . \quad (102)$$

and

$$\frac{\rho'}{\rho_c} = [\phi(b_1)]^3 \quad . \quad . \quad . \quad . \quad (103)$$

For the central temperature, from

$$\frac{1}{3}aT_c^4 \frac{\beta}{1-\beta} = K\rho_c^{\frac{4}{3}}$$

we find

$$T_c = \frac{(R/\mu)^{\frac{1}{3}}}{(\frac{1}{3}a)^{\frac{1}{3}}K} \left(\frac{1-\beta}{\beta}\right)^{\frac{1}{3}} \frac{1}{[\phi(b_1)]^{\frac{4}{3}}} \quad . \quad . \quad . \quad (104)$$

and

$$\frac{T'}{T_c} = [\phi(b_1)]^{\frac{4}{3}} \quad . \quad . \quad . \quad . \quad (105)$$

The radius r' of the core is given by (98) or (98′). The radius r_1 of the whole configuration is given by

$$\frac{r'}{r_1} = \frac{\xi'}{\xi_1} = \frac{A\xi'}{A\xi_1} = \frac{a_1}{\xi_0} \quad . \quad . \quad . \quad (106)$$

whence

$$r_1 = \frac{(\frac{1}{3}a)^{\frac{1}{2}}}{(\pi G)^{\frac{1}{2}}} \frac{K}{(R/\mu)^{\frac{3}{2}}} \frac{\xi_0 f(a_1)}{\beta^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (107)$$

The effective temperature T_e is then found to be given by

$$T_e = \frac{G^{\frac{1}{2}}(R/\mu)^{\frac{1}{2}}}{(\frac{1}{3}a)^{\frac{1}{12}}K^{\frac{1}{2}}} \left(\frac{L}{ac}\right)^{\frac{1}{4}} \frac{\beta^{\frac{1}{8}}(1-\beta)^{\frac{1}{12}}}{[\xi_0 f(a_1)]^{\frac{1}{2}}} \quad . \quad . \quad . \quad (108)$$

The ratio of the mass of the core to the mass of the whole configuration is given by

$$\frac{M(r')}{M} = \frac{A^2\xi'^2 f'(A\xi')}{A^2\xi_1^2 f'(A\xi_1)} = \frac{a_1^2 f'(a_1)}{\xi_0^2 f'(\xi_0)} \quad . \quad . \quad (109)$$

The mean density ρ_m from $\frac{4}{3}\pi r_1^3 \rho_m = M$ is given by

$$\rho_m = \frac{(\pi G)^3 (R/\mu)^2 M \beta (1-\beta)^{\frac{1}{2}}}{\frac{4}{3}\pi(\frac{1}{3}a)^{\frac{1}{2}}K^3\xi_0^3[f(a_1)]^3} \quad . \quad . \quad . \quad (110)$$

whence

$$\frac{\rho_m}{\rho_c} = \frac{3}{4} \frac{(\frac{1}{3}\pi a G^3)^{\frac{1}{2}}}{(R/\mu)^2} \frac{\beta^2}{(1-\beta)^{\frac{1}{2}}} M \frac{[\phi(b_1)]^3}{\xi_0^3[f(a_1)]^3} \quad . \quad . \quad (111)$$

In all these formulæ β stands for $1 - \kappa L/4\pi cGM$.

In expression (111) we recognise as a coefficient our old friend the discriminant C. Eliminating it by (94) and making free use of (100) and (101) we find that (111) can be written in the form

$$\frac{\rho_m}{\rho_c} = -\frac{3}{b_1}\phi'(b_1)\left[\frac{a_1 f'(\xi_0)}{\xi_0 f'(a_1)}\right] \quad . \quad . \quad . \quad (111')$$

30. *Case* (1). *f is of Collapsed Type,* ($L_0 > L > 0$), *ϕ is Emden's Solution of $n = \frac{3}{2}$. (Refined Theory of White Dwarfs.)*—Here ϕ is identical with the ϕ of Section C, paragraphs 21, 24; f depends on L. As L decreases below L_0, collapse proceeds further and further, the degenerate core gains at the expense of the gaseous envelope, and our formulæ must ultimately pass into the formulæ of paragraph 21 for the completely degenerate configuration. Thus as L decreases indefinitely we must have $a_1 \to \xi_0$, $b_1 \to \eta_0$.* Comparing (56) and (111′) we see that ρ_m/ρ_c for the composite model (111′) passes exactly into the form (56) as $a_1 \to \xi_0$, $b_1 \to \eta_0$. This shows that the composite configurations corresponding to f collapsed, ϕ Emden, may be identified with the white dwarfs consisting of a gaseous envelope surrounding a degenerate core. At the one end of the series ($L \to 0$) we get a completely degenerate "$n = \frac{3}{2}$" Emden polytrope. At the other end of the series the configurations pass continuously, as $L \to L_0 - 0$, into a complete Emden "$n = 3$" polytrope

* It is readily verified that $a_1 = \xi_0$, $b_1 = \eta_0$ are in fact solutions of (100) and (101) for L = 0. For then $\varphi(\eta_0) = 0$ and $f'(\xi_0) = \infty$, $f(\xi_0) = 0$.

with central density just equal to the critical density ρ'. By (103) and (105) we see, in fact, that as $L \to L_0 - 0$, $b_1 \to 0$, $a_1 \to 0$, and $f(\xi) \to$ Emden's solution of "$n = 3$."

The configurations may be illustrated in the following figures :—

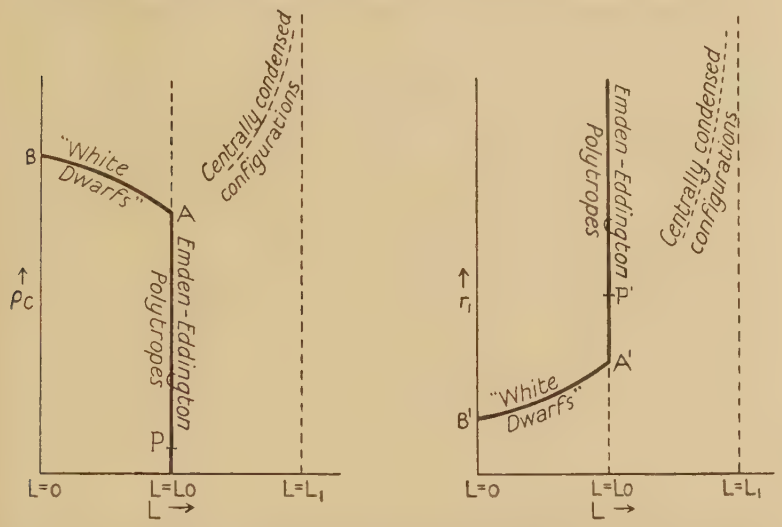

FIG. 4.—Linear series of configurations on the standard model for constant M and κ, for varying L. The Emden-Eddington polytropes are gaseous. The collapsed and centrally condensed configurations are composite ; they consist of a perfect-gas envelope surrounding a degenerate core, the latter possibly surrounding in turn a homogeneous incompressible core. The collapsed configurations AB, A′B′ are identified with "white dwarfs." The centrally condensed configurations have not been fully studied, but along them $r_1 \to \infty$ as $L \to L_1$, and as $L \to L_1, \varrho_c$ increases until it equals ϱ_{max}, the maximum density of matter. They may be identified with giants and ordinary dwarfs. Whether the linear series of centrally condensed configurations joins continuously on to AB has not been ascertained.

By use of the discriminant equation (94) to eliminate powers of R/μ and by frequent use of (100) and (101), all our formulæ may be thrown into forms which pass, as $a_1 \to \xi_0$, $b_1 \to \eta_0$, into those of Section C. It will be sufficient to enumerate them :

$$\rho_c = \frac{32\pi G^3 M^2 \beta^3}{125 K^3}\left[\frac{a_1^2 f'(a_1)}{\xi_0^2 f'(\xi_0)}\right]^2 \frac{1}{b_1^4[\phi'(b_1)]^2} \quad . \quad . \quad . \quad (102')$$

$$T_c = \left(\frac{8}{5}\right)^{\frac{1}{3}} \frac{G^{\frac{1}{3}}\pi^{\frac{2}{3}} M^{\frac{1}{3}} \beta(1-\beta)^{\frac{1}{3}}}{4^{\frac{1}{3}}(\frac{1}{3}a)^{\frac{1}{3}} K}\left[-\frac{a_1^2 f'(a_1)}{\xi_0^2 f'(\xi_0)} \frac{1}{b_1^2 \phi'(b_1)}\right]^{\frac{2}{3}} \quad . \quad (104')$$

$$r_1 = \frac{5K}{2^{\frac{1}{3}}\pi^{\frac{2}{3}} G M^{\frac{1}{3}} \beta}\left[\frac{\xi_0^2 f'(\xi_0)}{a_1^2 f'(a_1)}\right]^{\frac{1}{3}}[-b_1^2\phi'(b_1)]^{\frac{1}{3}}\frac{\xi_0}{a_1} b_1 \quad . \quad . \quad (107')$$

$$T_e = \frac{2^{\frac{1}{3}}\pi^{\frac{1}{2}} G^{\frac{1}{2}} M^{\frac{1}{6}} L^{\frac{1}{3}} \beta^{\frac{1}{2}}}{(ac)^{\frac{1}{4}}(5K)^{\frac{1}{2}}}\left[-\frac{a_1^2 f'(a_1)}{\xi_0^2 f'(\xi_0)} \frac{1}{b_1^5 \phi'(b_1)}\right]^{\frac{1}{6}} \quad . \quad . \quad (108')$$

These formulæ give exactly the same values as (102), (104), (107), (108), and they pass continuously into formulæ (55), (59), (61), (62) respectively on letting degeneracy approach completion ($L \to 0$), on remembering that β stands for $1 - \kappa L/4\pi cGM$.

We have now considered all the configurations which arise when $\psi = \phi(\eta)$ is Emden's solution of "$n = \frac{3}{2}$." We see that we have thus enumerated all the composite configurations for $L_0 > L > 0$. In fig. 4 the points A, A′ are the points of bifurcation, where $a_1 = 0$, $b_1 = 0$, and the series of Emden-Eddington ($n = 3$) polytropes ($L = L_0$) of arbitrary density and radius join the composite collapsed configurations of definite density and radius. (I have sketched in the linear series AB, A′B′ by eye.) We see at a glance that a configuration of Emden-Eddington type represented by a point such as P is unstable with regard to changes in L. A is the only stable "$n = 3$" polytrope. So far the panels $L_1 > L > L_0$ are untenanted. We anticipate that they will be occupied by one or more linear series corresponding to giants and ordinary dwarfs.

Case (2).*f of Centrally-condensed or Collapsed Type, ϕ a Fowler Solution of Collapsed Type.*—In this case, by the investigations of Section C, paragraph 23, the centre must be occupied by a region of still another equation of state or the configuration would only exist with an artificial supporting surface. For definiteness we now seek a solution in which the configuration consists of (*a*) a gaseous envelope, (*b*) a degenerate zone, (*c*) a homogeneous core of density $\rho = \rho_{max}$. We now have a second interface occurring, say, where $\eta = \eta''$. By the investigations of Section C, paragraph 23, η'' is the root of the equation

$$\tfrac{1}{3}\eta''^3[\psi(\eta'')]^{\frac{3}{2}} = -\eta''^2\left(\frac{d\psi}{d\eta}\right)_{\eta=\eta''} \quad . \quad . \quad . \quad (112)$$

where $\psi(\eta) = B^4\phi(B\eta)$. Put $B\eta'' = b_2$. Then (112) gives

$$\tfrac{1}{3}b_2^3[\phi(b_2)]^{\frac{3}{2}} = -b_2^2\phi'(b_2) \quad . \quad . \quad . \quad (113)$$

b_2 is a root of this equation, but the particular solution $\phi(\eta)$ (of "$n = \frac{3}{2}$") is not yet definite. Now ρ must be continuous at the interface and so $\rho_{max} = \rho''$ or

$$\rho_{max} = \lambda_2[\psi(\eta'')]^{\frac{3}{2}} = \lambda_2 B^6[\phi(b_2)]^{\frac{3}{2}},$$

or, since $\lambda_2 B^6$ is determined by (99),

$$\rho_{max} = \frac{(R/\mu)^4}{\frac{1}{3}aK^3}\frac{1-\beta}{\beta}\left[\frac{\phi(b_2)}{\phi(b_1)}\right]^{\frac{3}{2}} \quad . \quad . \quad . \quad (114)$$

Since $\rho_{max} > \rho'$, we must have $\phi(b_2) > \phi(b_1)$, and hence b_2 occurs inside b_1 on the upward curve of $\psi = \phi(\eta)$. *Cf.* fig. 3. Equations (94), (100), (101) hold as before. We can now see that the solution is determinate. Determine $f(\xi)$ to satisfy (94). It will be of the centrally-condensed type or collapsed type, according as $L_1 > L > L_0$ or $L_0 > L > 0$. Select an arbitrary $\phi(\eta)$ of Fowler type (2), (§ 23). Determine b_2 by solving (113), a_1 and b_1 by solving (100) and (101), and insert for b_1 and b_2 in

(114). We must suppose $\rho_{\max}$ to be given by known physical properties of the material such as the radii of electrons and nuclei. In general, the value of $\rho_{\max}$ computed from (114) will not agree with this value. By trial and error a $\phi(\eta)$ of Fowler type (2) must now be chosen so that $\rho_{\max}$, as computed from (114), agrees with the desired physical quantity. We then have a definite configuration.

(It is desirable to show that the condition implied by (114) as restricting $\phi(\eta)$ passes continuously into the similar condition (75) of § 23, as complete degeneracy sets in. By eliminating R/μ between (114) and the discriminant equation (94) and making free use of (100) and (101), we find that (114) is equivalent to

$$\rho_{\max} = \frac{32\pi G^3 M^2 \beta^3}{125 K^3}\left[\frac{a_1^2 f'(a_1)}{\xi_0^2 f'(\xi_0)}\right]^2 \frac{[\phi(b_2)]^{\frac{3}{2}}}{[b_1^2\phi'(b_1)]^2} \quad . \quad (114')$$

which passes continuously into (75) as $a_1 \to \xi_0$, $b_1 \to \eta_0$, $b_2 \to \sigma$.)

The temperature-structure of the homogeneous core of density $\rho_{\max}$ is readily found. The equation governing T is

$$\frac{dp'}{dr} = -\frac{\kappa L(r)}{4\pi c}\frac{\rho_{\max}}{r^2},$$

where

$$L(r) = \frac{L}{M}M(r) = \frac{L}{M}\tfrac{4}{3}\pi r^3\rho_{\max}.$$

Thus

$$\tfrac{1}{3}a\frac{dT^4}{dr} = -(1-\beta)G\tfrac{4}{3}\pi r\rho^2_{\max},$$

whence on integration

$$\tfrac{1}{3}a(T_c^4 - T''^4) = \tfrac{2}{3}\pi G(1-\beta)r''^2\rho^2_{\max}$$

or

$$\frac{T_c^4}{T''^4} = 1 + \frac{\frac{2}{3}\pi G(1-\beta)\rho^2_{\max}}{\frac{1}{3}a}\frac{r''^2}{T''^4}.$$

Now

$$\frac{r''}{r'} = \frac{\eta''}{\eta'} = \frac{B\eta''}{B\eta'} = \frac{b_2}{b'}$$

and r' is given by (98) or (98′); also

$$\frac{T''^4}{T'^4} = \left(\frac{\rho''}{\rho'}\right)^{\frac{4}{3}} = \left(\frac{\rho_{\max}}{\rho'}\right)^{\frac{4}{3}} = \left[\frac{\psi(\eta'')}{\psi(\eta')}\right]^{\frac{3}{2}} = \left[\frac{\phi(b_2)}{\phi(b_1)}\right]^{\frac{3}{2}}$$

and T′ is given by (81). After some reduction we find, using (114),

$$\frac{T_c^4}{T''^4} = 1 + \frac{5}{12}b_2^2[\phi(b_2)]^{\frac{1}{3}},$$

whence

$$\frac{T_c^4}{T'^4} = \left[1 + \frac{5}{12}b_2^2[\phi(b_2)]^{\frac{1}{3}}\right]\left[\frac{\phi(b_2)}{\phi(b_1)}\right]^{\frac{3}{2}} \quad . \quad . \quad (115)$$

This evaluates the central temperature when the configuration possesses a homogeneous core inside a degenerate zone, the whole inside a gaseous envelope. The three zones, though all "gases" in different states, may be likened to gaseous, liquid, and solid phases in some respects.

It appears that for L sufficiently near L_1 the configurations must possess an inner core of density ρ_{max}. For otherwise, since $\rho_c > \rho'$ and $\rho' \to \infty$ as $L \to L_1$, ρ_c would ultimately exceed ρ_{max}.

31. *The Radius. Giant Nature of very Luminous Configurations.*—The detailed discussion of the various formulæ we have secured demands a paper to itself.* I propose here to mention two points only, namely, the formula for the external radius (107), and the formula for the effective temperature T_e (108). Though (107) was specifically derived only for composite collapsed configurations, it holds generally whatever the types of the functions f and ϕ, provided only (100) and (101) can be solved to give a_1 and b_1. For a collapsed configuration (white dwarf type) $\xi_0 f(a_1)$ is a number comparable with unity, say not exceeding 6. But for the undiscussed centrally-condensed configurations a_1 will necessarily be very small and accordingly $\xi_0 f(a_1)$ very large, of the order of $1/a_1$. Consequently r_1 is very large; moreover, owing to the factor $\beta^{\frac{1}{3}}$ in the denominator, $r_1 \to \infty$ as $\beta \to 0$, *i.e.* as $L \to L_1$. We thus see that "giant" stars are predicted for $L \to L_1$. The order of magnitude of r_1 is seen by putting in the constants in (107); taking $\mu = 2m_H$, we find

$$r_1 = 2{\cdot}13 \times 10^8 \frac{\xi_0 f(a_1)}{\beta^{\frac{1}{3}}(1-\beta)^{\frac{1}{6}}} \text{ cms.} \quad . \quad . \quad (116)$$

This formula gives the radius of any composite configuration (partly gaseous, partly degenerate, including configurations with inner homogeneous cores) built on the standard model. It agrees of course with the order of magnitude of r_1 found in § 24 for a white dwarf when a_1 is evaluated for a highly collapsed configuration. See equations (107′) and (61).

32. *Division into Giants and Ordinary Dwarfs.*—The other application is to effective temperatures. From (108) we have, *apart from the factor* $[\xi_0 f(a_1)]^{-\frac{1}{2}}$,

$$T_e^4 \propto L\beta^{\frac{2}{3}}(1-\beta)^{\frac{1}{3}}$$

$$\propto L\left(1 - \frac{\kappa L}{4\pi c G M}\right)^{\frac{2}{3}}\left(\frac{\kappa L}{4\pi c G M}\right)^{\frac{1}{3}} \quad . \quad . \quad (117)$$

I am not in a position to discuss the variation of $[\xi_0 f(a_1)]^{-\frac{1}{2}}$ since this goes very deep into the structure of $f(\xi)$ as a function of L, but the other factors in T_e^4 of themselves suggest a division into two classes of configurations, corresponding to giants and ordinary dwarfs. For, from (117), T_e will have a maximum for some value of L (for given M), and for any given T_e less than $(T_e)_{max}$ there will be two values of L for given M. More definitely, suppose *for the first time* we adopt a mass-luminosity law (depending on the intrinsic physics of the energy-

* Besides tables of $f(\xi)$ and $\varphi(\eta)$ of various types, not yet calculated.

generating process) of the form $L = \alpha M^3$ (*cf.* Jeans, *Astronomy and Cosmogony*, p. 126). Then, by (117),

$$T_e^4 \propto M^{\frac{11}{3}}\left(1 - \frac{\kappa\alpha M^2}{4\pi cG}\right)^{\frac{2}{3}}.$$

T_e takes a *maximum* when

$$\frac{\alpha\kappa M^2}{4\pi cG} = \frac{11}{15},$$

or

$$\frac{\kappa L}{4\pi cGM} = \frac{11}{15},$$

or

$$L = \frac{11}{15}L_1.$$

Then

$$\frac{T_e^4}{(T_e)^4_{\max}} = \left(\frac{15}{11}\right)^{\frac{11}{3}}\left(\frac{15}{4}\right)^{\frac{4}{3}}\left(\frac{L}{L_1}\right)^{\frac{11}{3}}\left(1 - \frac{L}{L_1}\right)^{\frac{4}{3}} \quad . \quad . \quad (117')$$

For any value of T_e less than $(T_e)_{\max}$ there are thus two values of L/L_1; when $T_e/(T_e)_{\max}$ is small, one value of L/L_1 is small and one is near unity. This suggests the observed division into giants and ordinary dwarfs. The discussion is to be regarded as provisional until the behaviour of $f(a_1)$ is studied, but it is highly suggestive.

As regards the numerical value predicted for the effective temperature, we notice from (108) that it depends on β through the factor $\beta^{\frac{1}{6}}(1 - \beta)^{\frac{1}{12}}$. This is a slowly varying function of β, whose behaviour we have in fact just discussed, with its apparent control of the giant-dwarf classification, but its numerical value is always less than its maximum $(\frac{2}{3})^{\frac{1}{6}}(\frac{1}{3})^{\frac{1}{12}} = 0{\cdot}853$. With $\mu = 2m_H$, $L_\odot = 3{\cdot}78 \times 10^{33}$, (108) will be found to reduce to

$$T_e = \frac{88{,}200}{[\xi_0 f(a_1)]^{\frac{1}{6}}}\,\frac{\beta^{\frac{1}{6}}(1 - \beta)^{\frac{1}{12}}}{0{\cdot}853}\left(\frac{L}{L_\odot}\right)^{\frac{1}{4}} \text{ degrees} \quad . \quad (118)$$

This formula, like (116), gives by comparison with observation some idea of the magnitude of $f(a_1)$, and hence, from Fairclough's tables for which $\xi_0 = 1$, some idea of the order of a_1, and so of the radius and mass of the degenerate core, assuming a star to be built on the standard model. But I do not propose to make such estimates here. Lastly (118) accounts generally for the observed confined range of effective temperature, in spite of the huge range of variation of L.

These results concerning effective temperature should be contrasted with the so-called "luminosity formula" of the *current* theory, $L \propto M^{\frac{3}{2}}(1 - \beta)^{\frac{1}{2}}T_e^{\frac{4}{3}}$. This formula has been currently interpreted as implying a dependence of luminosity on effective temperature, but for a star in a steady-state this is an absurdity. The effective temperature must be a consequence of the total generation of energy—it cannot control it. If we invert the formula and write it $T_e \propto L^{\frac{3}{4}}M^{-\frac{3}{4}}(1 - \beta)^{-\frac{1}{6}}$, it fails to account in any way for the observed behaviour of T_e as

a function of L and M, as is well known. Further, it fails completely to account for the observed order of magnitude of T_e in actual stars.

Conclusion.

33. In *M.N.*, **90**, 17, 1929, I came to the conclusion that "it is not possible to infer from the observed masses, luminosities, and temperatures that the interiors of stars are necessarily composed of perfect gas; and that it is not possible to deduce the value of the absorption coefficient for the stellar interior.* Instead, we are led to infer a single definite fact concerning the internal density-distribution." This fact was the value of a certain pure number C, here called the discriminant. I pointed out that further research should indicate the nature of the restriction on the density-distribution imposed by the value of C.

The present paper justifies the above conclusions. We have seen by analysing particular source- and opacity-distributions by the method of integration inwards from the gaseous outer layers, that the value of C determines the steady-state configurations as being in general of two types, the centrally-condensed and collapsed, each comprising an outer gaseous envelope surrounding a degenerate core, which, in turn, may surround an incompressible core; the two types correspond to two ranges of luminosity separated by a critical value, for which luminosity the configurations are of the form of diffuse gaseous configurations. The last-named are shown to be in general unstable with respect to small changes in luminosity. The collapsed configurations reproduce the observed features of dense stars, the "white dwarfs" of nature. The centrally-condensed configurations may be provisionally identified with the giant stars and ordinary dwarf stars of nature. The physical properties of centrally-condensed distributions have been worked out in detail, but their numerical evaluation awaits the construction of certain tables.

Summary.

1. The mass M, luminosity L, and opacity κ of a star must be taken as independent variables so far as steady-state considerations are concerned.

2. The observed correlation of L with M in nature must depend upon the intrinsic physics of the energy-generating process and cannot be deduced from steady-state considerations only.

3. The basic problem of stellar structure is: What are the steady-

* The argument for this conclusion given in my earlier paper arose out of the circumstance that, at best, analysis of steady-state configurations *via* C can only determine the march of the product $\kappa(r)L(r)$ through the interior. Separation of $\kappa(r)$ and $L(r)$ requires the introduction of the physics of energy-generation. Thus, although *for a given model* knowledge of the density (or r_1 or T_e), combined with knowledge of L and M, fixes κ in principle, *e.g.* through formulæ (110), (107), or (108), the determination of the model on which an actual star is built will require the physics of energy-generation. I hope on a future occasion to give the theory of models of non-constant opacity.

state configurations of a given mass M, of given opacity, evolving energy at the rate L, as L varies from zero upwards, M remaining constant ?

4. The method of analysis of this problem is to integrate the equations expressing the steady state, from the boundary inwards. The data of the problem, for any given relative source-distribution and any given opacity law, provide just sufficient boundary conditions to effect the integration. The integration continues inwards with the perfect-gas equation of state until the solution shows that the gas-laws cease to be obeyed.

5. The analysis is carried out in detail for two special source- and opacity-distributions, namely, the point-source model and the standard model (ϵ = constant), both of constant opacity. It is shown that for each of these the following results hold. There are two critical luminosities, L_1 and L_0, such that for $L > L_1$ no steady-state configurations exist ; for $L_1 > L > L_0$ the gas-laws break down because the density ρ would otherwise increase indefinitely ; for $L_0 > L > 0$ the gas-laws break down because radiation-pressure is too small for a perfect-gas diffuse distribution to sustain its own weight. For $L = L_0$ diffuse perfect-gas distributions are possible of arbitrary radius. Configurations $L_1 > L > L_0$ are described as " centrally-condensed " and correspond to the giants and ordinary dwarfs of nature ; configurations $L_0 > L > 0$ are described as " collapsed " and correspond to the dense stars (" white dwarfs ") of nature.

6. The properties of " highly-collapsed " configurations consisting of completely degenerate material on the standard model are worked out in detail. The central density for $M = 0{\cdot}85\ \odot$ is of the order of 3×10^6 gram cm.$^{-3}$, the radius 10^9 cms., the effective temperature 11,000°. The central temperature is of the order of $10^9(\kappa L/4\pi cGM)^{\frac{1}{3}}$ degrees, the evaluation of this needing a knowledge of the opacity of a degenerate gas. The mean density is about $\frac{1}{10}$ of the central density. As $L \to 0$, density and radius tend to definite limits (functions of M) and the temperature decreases indefinitely.

7. The properties of composite configurations consisting of a gaseous envelope surrounding a degenerate core (the latter possibly surrounding, in turn, a homogeneous core) are worked out in detail for the standard model. The density in the transition region between envelope and core is of the order of $4 \times 10^7[\kappa L/(4\pi cGM - \kappa L)]$ gram cm.$^{-3}$, the temperature of the order of $10^{10}[\kappa L/(4\pi cGM - \kappa L)^{\frac{1}{3}}]$ degrees. Both increase indefinitely as L approaches $L_1 (= 4\pi cGM/\kappa)$. Inside the core the density and temperature are higher still. Formulæ are given for the radius and mass of the core, and the external radius of the configuration, but their evaluation depends on the solution of certain transcendental equations ; these, in turn, require the construction of tables of solutions of Emden's differential equations not yet completed. Giant stars may contain an inner core of the maximum density of matter. As $L \to L_1$, the external radius $\to \infty$. Thus very luminous configurations have the " giant " property, namely, large radius.

8. Composite collapsed configurations form a linear series. Emden-Eddington polytropes form another linear series which intersects the

former in a configuration of diffuse type whose central density is so large that degeneracy is just setting in. Composite centrally-condensed configurations presumably form one or more other linear series, but owing to analytical difficulties their relations to the former linear series have not been worked out in detail. It is possible that the passage of a configuration through the critical value $L = L_0$ may be discontinuous, as it certainly is for perfect-gas configurations of unlimited compressibility. If subsequent analysis confirms this, the passage of a configuration through $L = L_0$ would exhibit the phenomena of a nova, and we should have the suggestion that every star passes through a nova stage as it crosses from a configuration of "ordinary density" to one of great density, as its evolution of energy decays.

9. The physical states of stars thus being implied by discussion of steady-state configurations, the way is prepared for a discussion of the physics of energy-generation and the reason for the mass-luminosity correlation observed in nature. Roughly speaking, the cause of this correlation is that the more massive the star the greater is the amount of mass at a temperature high enough for the generation of energy out of matter. The cores of very luminous configurations are found to be hotter than those of less luminous configurations. Again, the temperature of the core increases with increasing opacity. Both these conclusions are in accordance with common-sense. On the current theory the central temperature, on the other hand, is independent of luminosity and opacity, and depends only on mass.

10. All formulæ for the physical characteristics of a star are stated in terms of the three formally independent variables M, L, and opacity. No formula exists connecting M, L, and opacity from steady-state considerations only.

11. The method of the present paper is applied towards elucidating the possible steady-state configurations that could exist. It could equally well be applied to find the state of any actual star by numerical integration from the boundary inwards.

12. A theory of the origin of stellar energy and the mechanism of the conversion of matter into radiation is outlined (§ 1). The conversion is promoted by the natural tendency of a star to cool and is thus non-explosive. The whole process is analogous to the cooling of a body of high specific heat.

13. The present paper is merely a first survey of the problems that arise. Further research is required to investigate (1) the effect of relativity corrections on equations of state and dissociation formulæ at the high temperatures determined; (2) the effect of electrostatic corrections at the high densities determined; (3) the maximum density of which ionised matter is capable; (4) the gravitational effect of the radiant energy at the high temperatures determined; (5) the opacity of a degenerate gas; (6) the effect of other opacity-laws than κ = constant; * (7) the possible effect of a state of dissociative equili-

* I have examined the law $\kappa \propto \varrho/T^{7/2}$. The equations are found to possess singularities at $r = 0$ of much the same type as for κ = constant.

brium between nuclei, electrons, and radiation in the hot dense core.* It would be desirable to give a complete formal solution for each of these factors rather than to attempt rough calculations.

It is my pleasant duty to express my thanks to those investigators who have so kindly assisted in these researches. Mr. Norman Fairclough performed a numerical integration of Emden's "$n = 3$" equation, working from the boundary inwards, at a crucial stage of the investigation. His results influenced much of the course of the argument from that point onwards. For Mr. Fairclough's services I am indebted to a grant from the Department of Scientific and Industrial Research. To Mr. T. G. Cowling I am indebted for theoretical considerations which indicated that it was the more luminous configurations which might be expected to possess central condensations, and not the less luminous. To Mr. R. H. Fowler I am indebted for exhaustive analysis of the properties of solutions of the differential equations that arose. His work lifted my results to a level of rigour quite beyond my own powers. I am especially grateful to him for pointing out to me the existence of a type of solution of Emden's "$n = \frac{3}{2}$" equation which (though searching for it) I had entirely overlooked.

Numerical Integration of Emden's Polytropic Equation of Index Three.

By N. Fairclough, M.Sc.

(*Communicated by Professor E. A. Milne.*)

Numerical integration of the equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \cdot \frac{d\theta}{d\xi}\right) = -\theta^3 \quad . \quad . \quad . \quad (1)$$

may be carried out in several ways, provided we know an initial set of values for ξ, θ, and $d\theta/d\xi$. In the following tables, which are provisional integrations, the object has been to present numerical integrations of equation (1) with the initial conditions:—

at $\xi = 1$, $\theta = 0$, $d\theta/d\xi =$ a given negative number;

and, accordingly, to determine θ, and $d\theta/d\xi$ from $\xi = 1$ to $\xi = 0$.

If we proceed with the numerical solution of equation (1), *outwards* from $\xi = 0$, $\theta = 1$, $d\theta/d\xi = 0$, and arrive at a solution (Emden's solution),

$$\theta = f(\xi),$$

then,

$$\theta = Af(A\xi)$$

is also a solution, where A is arbitrary.

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* Later calculations indicate that the mass of the core may consist almost entirely of the mass of the imprisoned radiant energy. This reservoir of radiant energy provides the actual energy which the star radiates to space, and is itself fed by the conversion of matter.

Therefore, if $f(\xi)$ vanishes at $\xi_0 (= 6{\cdot}8969)$, then

$$\theta = \mathrm{A}f(\mathrm{A}\xi)$$

will vanish at $\xi = \xi_1$, where $\mathrm{A}\xi_1 = \xi_0$. By taking $\mathrm{A} = \xi_0$ we have a solution vanishing at $\xi = 1$. For any value of A,

$$\left[\xi^2 \frac{d\theta}{d\xi}\right]_{\xi=\xi_1} = [\xi^2 \mathrm{A}^2 f'(\mathrm{A}\xi)]_{\xi=\xi_1} = \xi_0^2 f'(\xi_0) = -2{\cdot}01813.$$

Thus, $\xi^2 \frac{d\theta}{d\xi}$ at the boundary is the same for all values of A, and is equal to $-2{\cdot}01813$. Accordingly, the solution of equation (1) computed for

$$\xi = 0, \qquad \theta = 1, \qquad \frac{d\theta}{d\xi} = 0,$$

may be converted into the solution for which

$$\xi = 1, \qquad \theta = 0, \qquad \left(\frac{d\theta}{d\xi}\right)_{\xi=1} = -2{\cdot}01813$$

by dividing all abscissæ by 6·8969 and multiplying the corresponding ordinates by 6·8969. Table I. below shows this reduced critical solution. It could have been constructed from the table given by Emden (*Gaskugeln*, p. 80, Tabelle 7, "$n = 3$"), but in order to have all the work on the same footing Emden's solution was re-calculated and then reduced in the manner described so as to make it vanish at $\xi = 1$. The values differ slightly from Emden's.

Equation (1) was then integrated numerically working *inwards* from the boundary. Each solution was computed for the initial conditions

$$\xi = 1, \qquad \theta = 0, \qquad \left(\frac{d\theta}{d\xi}\right)_{\xi=1} = -a,$$

a being some definite positive number not coinciding with the critical value 2·01813. Tables II.–VII. give the solutions for a variety of values of a, selected so as to lie in pairs above and below the critical value. The value of $-\xi^2 d\theta/d\xi$ is tabulated in each case as well. It may be mentioned that in the stellar model, whose structure is represented by (1), θ is proportional to the temperature at a point at a fraction ξ of the total radius of the star from the centre; and the ratio of $-\xi^2 d\theta/d\xi$ at any point to its value at $\xi = 1$ is equal to the surviving fraction of the total mass remaining inside the point in question.

TABLE I.

(*Emden's solution, recalculated and reduced.*)

$\xi = 1, \quad \theta = 0, \quad d\theta/d\xi = -2{\cdot}01813.$

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
0	6·89690	0	0
·036248	6·82572	3·89052	·00511
·072496	6·61992	7·36531	·03871
·108744	6·30046	10·11231	·11958
·144993	5·89719	11·99645	·25220
·173991	5·53525	12·87787	·38985
·202990	5·15502	13·27316	·54692
·231988	4·76934	13·26935	·71415
·260987	4·38836	12·96302	·88297
·289985	4·01965	12·43930	1·04604
·326234	3·58356	11·59736	1·23429
·362482	3·18009	10·65363	1·39982
·398730	2·81145	9·68897	1·54044
·434978	2·47737	8·75475	1·65644
·471226	2·17611	7·88046	1·74984
·507474	1·90520	7·07991	1·82334
·543723	1·66188	6·35926	1·87996
·579971	1·44325	5·71615	1·92265
·616219	1·24662	5·14629	1·95411
·652467	1·06936	4·64304	1·97671
·688715	·909287	4·20066	1·99246
·724963	·764246	3·81109	2·00306
·761212	·632446	3·46860	2·00988
·797460	·512302	3·16703	2·01401
·833708	·402434	2·90065	2·01632
·869956	·301670	2·66567	2·01748
·906204	·208907	2·45732	2·01796
·942452	·123248	2·27181	2·01811
·978701	·043933	2·10675	2·01813
·999725	·000552	2·01923	2·01813
1·000000	0·0	2·01813	2·01813

TABLE II.*

$\xi = 1, \quad \theta = 0, \quad d\theta/d\xi = -2{\cdot}1192$ (*about 5 per cent. above the critical value*).

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
0	0	2·119200	2·119200
·954939	·1	2·323908	2·119190
·913762	·2	2·537912	2·119059
·875985	·3	2·760898	2·118578

* Table calculated by taking θ as independent variable, except for last five entries.

TABLE II.*—*continued.*

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
0·841196	0·4	2·992420	2·117466
·779220	·6	3·478895	2·112329
·725569	·8	3·992598	2·101904
·678546	1·0	4·528502	2·085037
·627161	1·25	5·221996	2·053972
·582268	1·50	5·932878	2·011459
·542492	1·75	6·651995	1·957668
·506806	2·0	7·370460	1·893120
·468290	2·3	8·219442	1·802487
·433512	2·6	9·039710	1·698854
·401687	2·9	9·815625	1·583774
·372202	3·2	10·53054	1·458840
·344559	3·5	11·16624	1·325664
·318338	3·8	11·70201	1·185870
·293166	4·1	12·11340	1·041100
·268687	4·4	12·37010	·893032
·244531	4·7	12·43244	·743405
·220263	5·0	12·24433	·594045
·195290	5·3	11·71792	·446898
·168623	5·6	10·69186	·304009
·138047	5·9	8·77083	·167145
·111751	6·1	6·26478	·078237
·091678	6·2	3·53597	·029719
·079525	6·23	1·30920	·008280
·073725	6·233906	·00574	·000031
·073701	6·233906	0·0	0·0
·07	6·23219	− ·94459	−·004629
·06	6·20776	− 4·11985	−·014831
·04	6·02162	−16·6224	−·026596

TABLE III.

$\xi = 1$, $\theta = 0$, $d\theta/d\xi = -1{\cdot}90$ (*about 5 per cent. below the critical value*).

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
1·00	0	1·900000	1·900000
·95	·100000	2·105251	1·899989
·90	·211106	2·345449	1·899813
·80	·474819	2·963641	1·896730
·70	·812410	3·840233	1·881714
·60	1·255563	5·099223	1·835720
·50	1·850275	6·900563	1·725141

* Table calculated by taking θ as independent variable, except for last five entries.

TABLE III.—*continued.*

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
0·40	2·658025	9·376077	1·500172
·35	3·163019	10·84355	1·328334
·30	3·743249	12·36546	1·112891
·25	4·397430	13·76520	·860325
·20	5·112929	14·76760	·590704
·15	5·862685	15·10351	·339829
·10	6·617502	15·14777	·151478
·08	6·925541	15·81647	·101225
·06	7·262490	18·41253	·066285
·04	7·707715	28·37981	·045408
·02	8·684494	89·56425	·035826
·01	10·42860	340·0568	·034006
·006	12·68638	934·3472	·033637
·002	24·44974	8348·575	·033394
·001	41·27798	33326·32	·033326
·0005	74·88603	133140·1	·033285
·0003	119·3700	369556·7	·033260
·0001	353·1502	3321491	·033215

TABLE IV.

$\xi = 1$, $\theta = 0$, $d\theta/d\xi = -0{\cdot}201828$ (*about* $\frac{1}{1\cdot 0}$ *of the critical value*).

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
1	0	·201828	·201828
·9	·022425	·249170	·201828
·8	·050457	·315350	·201824
·7	·086495	·411848	·201806
·6	·134538	·560416	·201750
·5	·201767	·806440	·201610
·4	·302499	1·258123	·201300
·3	·469983	2·229353	·200642
·2	·803210	4·980615	·199225
·1	1·790307	19·57362	·195736
·08	2·277956	30·37799	·194419
·06	3·084173	53·50945	·192634
·04	4·677911	118·7589	·190014
·03	6·253116	209·0228	·188121
·02	9·366041	463·6073	·185443
·01	18·51447	1809·512	·180951
·006	30·46290	4938·002	·177768
·002	88·50825	42829·94	·171320
·001	173·1373	167596	·167596
·0005	338·8038	656440	·164110

TABLE V.

$\xi = 1$, $\theta = 0$, $d\theta/d\xi = -20{\cdot}18277$ (*about* 10 *times the critical value*).

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
1	0	20·18277	20·18277
·95	1·062132	22·34833	20·16937
·90	2·237347	24·64209	19·96010
·85	3·516746	26·32791	19·02192
·80	4·833344	25·78711	16·50376
·75	6·018205	20·66929	11·62648
·70	6·793446	9·31828	4·56596
·675	6·931869	1·58142	0·72054
·67	6·935647	− 0·07463	− 0·03350
·65	6·866368	− 6·89130	− 2·91158
·60	6·100258	−23·4656	− 8·44760
·55	4·579416	−36·7715	−11·12338
·50	2·472857	−47·2841	−11·82101
·45	− 0·162636	−58·5993	−11·86635
·40	− 3·454307	−73·6331	−11·78129
·35	− 7·500136	−86·2111	−10·56086
·30	−11·50484	−64·8694	− 5·83825
·275	−12·70357	−27·8758	− 2·10811
·25	−12·76701	24·6912	1·54320
·20	− 8·81284	133·307	5·33228
·15	0·55156	252·211	5·67475
·10	18·7724	484·676	4·84676
·08	28·3101	421·924	2·70031
·06	32·6280	− 77·193	− 0·27790
·04	22·8782	− 1010·85	− 1·61736
·03	8·9503	− 1877·81	− 1·69003
·02	− 19·263	− 4217·38	− 1·68695
·01	− 96·345	−13727·2	− 1·37272
·005	− 188·93	−25789·2	− 0·64473
·001	− 331·24	− 128831	− 0·12883
·0005	− 451·24	− 456812	− 0·11420
·0001	− 1489·9	−10616500	− 0·10617

TABLE VI.

$\xi = 1$, $\theta = 0$, $d\theta/d\xi = -1{\cdot}009139$ (*about* $\frac{1}{2}$ *the critical value*).

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
1	0	1·009139	1·009139
·95	·053113	1·118157	1·009137
·90	·112126	1·245815	1·009111
·80	·252258	1·576013	1·008648
·70	·432209	2·053861	1·006392

TABLE VI.—*continued.*

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
0·60	1·671093	2·776163	·999419
·50	1·001588	3·928872	·982218
·40	1·483783	5·905707	·944913
·35	1·815752	7·459315	·913766
·30	2·240575	9·671532	·870438
·25	2·800875	12·97850	·811156
·20	3·571062	18·29046	·731619
·15	4·698767	27·90314	·627821
·10	6·551165	49·84640	·498464
·08	7·720349	68·87505	·440800
·06	9·420915	105·8616	·381102
·04	12·31117	200·6044	·320967
·02	19·42930	654·8125	·261925
·01	31·65386	2319·247	·231925
·005	53·93187	8589·548	·214739
·003	81·95871	22894·3	·206049
·001	219·1958	193125	·193125
·0005	409·9014	746408	·186602

TABLE VII.

$\xi = 1$, $\theta = 0$, $d\theta/d\xi = -4{\cdot}03626$ (*about twice the critical value*).

ξ.	θ.	$-(d\theta/d\xi)$.	$-(\xi^2 . d\theta/d\xi)$.
1·00	0	4·036260	4·036260
·95	·212434	4·472192	4·036153
·90	·448432	4·980830	4·034473
·85	·711919	5·573431	4·026802
·80	1·007309	6·257785	4·004982
·75	1·339229	7·033488	3·956337
·70	1·711916	7·883738	3·863032
·65	2·128095	8·762202	3·702031
·60	2·587053	9·573981	3·446633
·55	3·081623	10·151881	3·070945
·50	3·594017	10·235839	2·558960
·45	4·091070	9·473583	1·918401
·40	4·520578	7·467699	1·194832
·35	4·811422	3·877064	·474940
·314	4·887611	·195021	·019228
·312321	4·887775	0	0
·30	4·878678	− 1·498120	− ·134831
·28	4·822461	− 4·176978	− ·327475
·25	4·629385	− 8·837010	− ·552313
·22	4·282514	−14·49215	− ·701420

The relations between ξ and θ, as set out in the foregoing tables, are shown graphically in the following diagram (fig. 1). The curves are lettered from A to G, to correspond respectively to Tables I.–VII.

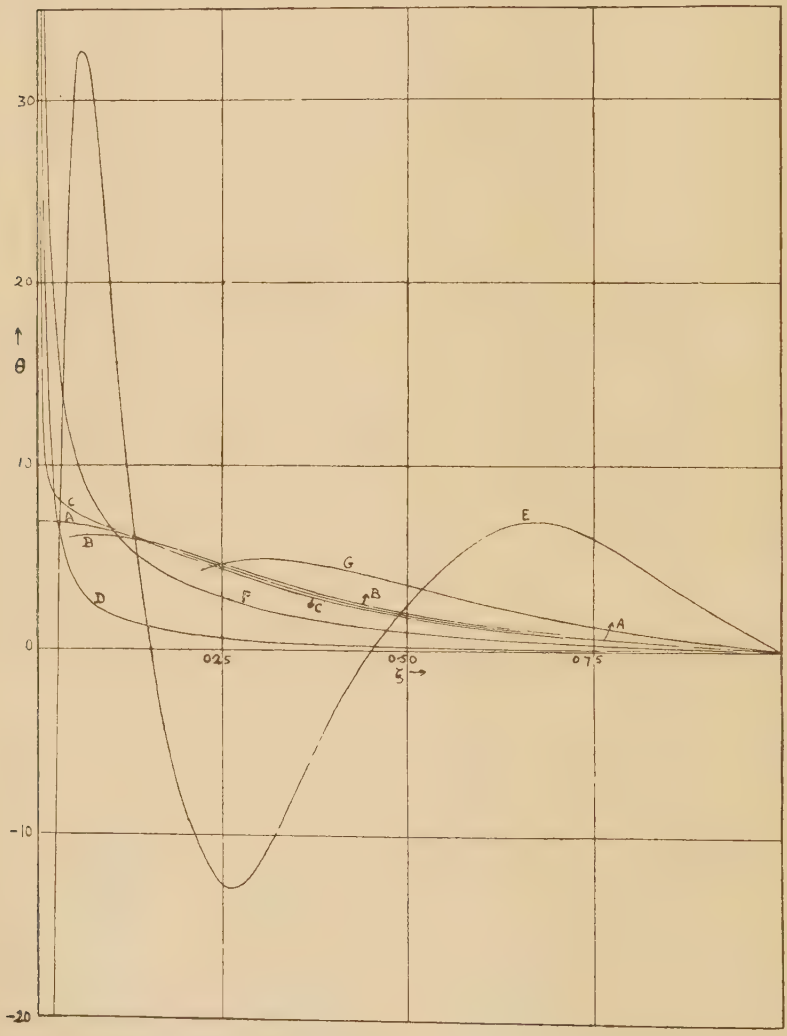

FIG. 1.—Integral curves of $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3$ passing through $\xi = 1$, $\theta = 0$, for different values of $(d\theta/d\xi)_{\xi=1}$.

I wish to acknowledge with thanks my indebtedness to the Department of Scientific and Industrial Research for the grant which has

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Newport, Mon.:
1930 Nov. 11.

The Solutions of Emden's and Similar Differential Equations.

By R. H. Fowler, F.R.S.

§ 1. *Introduction. Physical Bearings of the Problem.*—A sphere of perfect gas, in equilibrium under gravitation, gas-pressure and radiation-pressure, generating energy at a certain rate per unit mass, has been shown by Eddington on certain natural simplifications to have a structure controlled by the equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^3=0 \qquad . \quad . \quad . \quad (1)$$

In this equation ξ represents a " reduced " radius, and θ a " reduced " temperature. Other physical quantities follow at once when θ is known as a function of ξ. The equation is a member of the family of equations shown by Emden to govern the polytropic * equilibrium of a gravitating gas sphere, namely,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^n=0 \qquad . \quad . \quad . \quad (2)$$

Equation (2) for $n=\frac{3}{2}$ is allied to that governing the Thomas-Fermi distribution of electrons in an atom,† differing only in the sign of $\theta^{\frac{3}{2}}$.

The application of equation (1) and similar equations to astrophysical problems has recently been re-examined by Milne, to see whether it may be that any oversight has occurred such as might affect the conclusions drawn. Milne's investigations will be found on page 4 of this number, and it is not within my province to comment on them here, beyond what is essential to explain the origin of my own studies. In the great papers in which Eddington initiated the present theory of the internal structure of stars, Eddington bases his discussion on the properties of Emden's solution of the equation (1) which satisfies the central conditions

$$\theta=1, \qquad \frac{d\theta}{d\xi}=0 \qquad (\xi=0) \, . \qquad . \quad . \quad (3)$$

But $\xi=0$ is a singularity of the equation, and it may not be safe without further examination to base any discussion on the properties of such a particular solution, unless it is known *a priori* that this is the solution required. The reader of Milne's paper will see that it

* Polytropic equilibrium of a gas denotes a state of the gas in which by hypothesis $p=\varphi\rho^\gamma$; γ, φ constant.

† L. H. Thomas, *Proc. Camb. Phil. Soc.*, **23**, 542 (1926).

follows at once from Milne's arguments that for just this reason Eddington's theory may benefit by revision, or at least by reconsideration. In view, therefore, of the great importance of Milne's contentions, and so of the general character of *all* the solutions of Emden's and similar equations, it seems desirable to re-examine the character of the solutions of equation (1) or (2) as an exercise in pure mathematics. This I attempt to do in this paper, stimulated thereto, as is obvious, by many discussions with Professor Milne.

In undertaking this discussion it serves no useful purpose to restrict it unduly to equation (1), or even (2). The more general equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\xi^{\sigma'}\theta^n=0 \qquad (n>1) \quad . \quad . \quad (4)$$

might also be of value, and is about as easy to discuss.

Emden himself was well aware that the solution which he constructed numerically was a particular one, and that there were others with quite different properties as $\xi\to 0$, but he has not determined the properties of these other solutions—at least not in a sufficiently usable form. A rigorous though incomplete study of equations such as (4) has been given before, but has unfortunately remained unknown to astrophysicists. The study was started by Horn and Kneser, arranged as a discussion of the form as $x\to+\infty$ of real continuous solutions of equations such as

$$\frac{d}{dx}\left(x^\rho\frac{dy}{dx}\right)+bx^\sigma y^n=0 \quad . \quad . \quad . \quad (5)$$

and similar equations of higher order. It was restarted on rather different lines by Hardy * for the more tractable first-order equations

$$\mathrm{P}(x,y,y')=0 \quad . \quad . \quad . \quad . \quad (5a)$$

where P is a polynomial in x, y, y'. Finally, perhaps the most nearly complete existing discussion of equation (5) is my own,† at least so far as I am aware. Like everyone else I had entirely overlooked the fact that these discussions had astrophysical applications. The form of the more general solutions of equation (1) has, of course, been rediscovered, and still more their importance recognised for the first time by Milne. The values of n, considered in equation (5), have in all previous studies except Emden's been positive integers. It is comparatively simple to extend all the theorems to rational values of n greater than unity. The assumption that n is a positive integer is only relevant in controlling the behaviour of y^n when y changes sign. None of these studies tells the astrophysicist quite all he wants to know, and it is my object here to simplify and fill in the gaps, so that the behaviour of any solution having an astrophysical interest may be rigorously established.

* Hardy, *Proc. Lond. Math. Soc.*, **10**, 451 (1911).

† Fowler, *Quart. J. of Math.*, **45**, 289 (1914). Fairly full earlier references are given in these papers and will not be repeated here.

As I have said, all discussions have been aimed at obtaining the form of the solutions of the proposed differential equation as $x \to +\infty$, and it is frequently convenient to adhere to this practice. In astrophysical problems it is the form of the solutions as $\xi \to 0$ which is in doubt and of supreme importance to us. On writing

$$\xi = 1/x$$

equation (4) becomes

$$\theta'' + x^{-(4+\sigma')}\theta^n = 0.$$

We shall therefore here frequently discuss rather the equation

$$\theta'' + x^{\sigma}\theta^n = 0 \quad . \quad . \quad . \quad . \quad (6)$$

remembering that for astrophysical purposes $\sigma = -4 - \sigma'$ and that usually $\sigma = -4$.

We encounter certain special cases, of which one at least is more important than the general case, for it covers equation (1). This case is

$$\sigma + n + 1 = 0 \quad . \quad . \quad . \quad . \quad (7)$$

For example, equation (1) with $\sigma = -4$, $n = 3$ satisfies (7). We can then write

$$\theta = xy \qquad x = e^t \qquad (t \to \infty),$$

for the equation is homogeneous, and obtain

$$y'' + y' + y^n = 0 \quad . \quad . \quad . \quad . \quad (8)$$

Another reduction is possible when

$$\sigma + 2 = 0 \quad . \quad . \quad . \quad . \quad (9)$$

On putting

$$x = e^t \qquad (t \to \infty),$$

we have

$$\theta'' - \theta' + \theta^n = 0 \quad . \quad . \quad . \quad . \quad (10)$$

This equation is not of much interest to astrophysicists, but the interest of the special case (7) is extreme. We shall therefore start with its discussion.

We close this introduction with one remark on notation. Since it is convenient to pass freely to and fro between the independent variables ξ, x, t, and the dependent variables θ, y, we standardize the notation

$$\xi = 1/x = e^{-t} \qquad (\xi \to 0, t, x \to +\infty) \quad . \quad . \quad (11)$$

$$\theta = \frac{1}{\xi}y = xy = e^t y \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In writing differential coefficients we use y', etc., indiscriminately for $dy/d\mathrm{X}$, where X is whichever of the variables ξ, x, t we are working with at the moment.

§ 1.1. *Summary for Astrophysicists.*—For the sake of such astrophysicists as may be interested in the results of this paper, rather than in their derivation, I close this introduction with a short statement of the

results proved to hold for the two equations which are used extensively in Milne's work—namely,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^3=0 \quad . \quad . \quad . \quad \text{(I.)}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)+\theta^5=0 \quad . \quad . \quad . \quad \text{(II.)}$$

Equations I. and II.—(1) As $\xi \to 0$ all the solutions are ultimately monotonic functions of ξ and steadily approach either a finite limit or $\pm\infty$.

(2) Consider the group of all the solutions which start inwards from a given point (ξ_0, θ_0), $\theta_0 \geqslant 0$, with various values of the initial slope such that θ initially increases inwards. Starting-points ξ_0, θ_0 are divided into two classes by a simple *critical curve* of the form $\theta = \alpha\xi^{\varpi}$ (α a positive constant, $\varpi = -1$ for equation I., $\varpi = -4$ for equation II.).

(3) If the starting-point lies below the critical curve there pass through the starting-point two, and only two, solutions of the group which remain positive in the range 0, ξ_0 and tend to finite limits as $\xi \to 0$. These solutions are both simple transformations of Emden's solution, and are called E-solutions. If the starting-point lies above the critical curve no solution of the group is an E-solution, and if the starting-point lies on the curve there is just one member of the group which is an E-solution touching the critical curve at the starting-point. [The critical curve is the envelope of the E-solutions.]

(4) If the starting-point lies below the critical curve and the starting-slope lies between the slopes of the two E-solutions through the starting-point, then any such solution (called an M-solution) remains always positive as $\xi \to 0$, and in fact tends steadily to positive infinity with the asymptotic form

$$\theta \sim \frac{1}{\xi}\left[\frac{1}{2\log(C/\xi)}\right]^{\frac{1}{2}} \quad . \quad . \quad \text{(Equation I.)}$$

$$\theta \sim \frac{C}{\xi} \quad . \quad . \quad . \quad . \quad \text{(Equation II.)}$$

where C is some positive constant. It remains always below the critical curve.

(5) If the starting-point lies below the critical curve and the starting-slope is *less* than the *lesser* of the slopes of the two E-solutions, then there exists a number ξ_1, such that $0 < \xi_1 < \xi_0$, and $\theta(\xi_1) = 0$. The solution lies entirely below both E-solutions in the interval ξ_0, ξ_1.

(6) If the starting-point lies below the critical curve and the starting-slope is *greater* than the *greater* of the slopes of the two E-solutions, then the solution must pass above the critical curve.

(7) If the starting-point lies above the critical curve or if the solution ever rises above the critical curve as in (6), then there again exists a number ξ_1, such that $0 < \xi_1 < \xi_0$, $\theta(\xi_1) = 0$.

Only E-solutions and M-solutions are permissible solutions in any astrophysical applications over a range which reaches to $\xi = 0$, as all other solutions have illegitimate zeros. The establishment of the

asymptotic forms and relative arrangements of the E-solutions and the M-solutions constitutes the main result of this paper.

§ 2. *Study of the Equation* * $(n = \frac{p}{q} > 1\ ;\ p, q$ *odd*$)$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \xi^{n-3}\theta^n = 0, \qquad y'' + \frac{1}{\xi^2}y^n = 0,$$

$$\theta'' + x^{-n-1}\theta^n = 0, \qquad y'' + y' + y^n = 0. \qquad \text{(A)}$$

Part I. Possible forms as $t \to \infty$ *of the solutions of* $y'' + y' + y^n = 0$.— This equation is one of those not completely discussed before. Part I. of the discussion can be finished off by the proper appeal to Hardy's work, and is given here in a simplified form *de novo*.† After the results of Part I. are obtained, the complete properties of the solutions of (A) can be established by bringing in a different type of argument.

All variables are to be real throughout. We start by confining attention to solutions $y = y(t)$, which are such that $y(t)$, $y'(t)$ exist and are real, finite, and continuous for all finite $t(> t_0)$, without inquiring more closely whether any such solutions actually exist. Such solutions of course also have continuous $y''(t)$ and differential coefficients of all higher orders for all $t(> t_0)$, except perhaps at $y = 0$. They will be called *proper solutions* when a name is required. We interpret

$$(-y)^n = -(y^n).$$

We prove the following theorem :—

Theorem I.‡*—Any proper solution* $y = y(t)$ *of* (A) *is such that* $y(t) \to 0$ *with or without an infinity of zeros for* $t > t_0$.

For any positive k, however small, and any proper solution $y = y(t)$ there are three *a priori* possibilities :

(i) There exists a t_0 such that $y(t) > k \quad (t > t_0)$.
(ii) There exists a t_0 such that $y(t) < k \quad (t > t_0)$.
(iii) $y = y(t)$ cuts $y = k$ in a sequence of points $\{t_s\}$ such that $t_s \to \infty$.

We shall, for the sake of shortness, call any such sequence simply "a $\{t_s\}$".

Case (i) is impossible, for the hypothesis implies

$$\begin{aligned} y'' + y' &< -k' && (t > t_0), \\ \frac{d}{dt}(e^t y') &< -k'e^t, \\ e^t y' &< -k'e^t + C, \\ &< -k''e^t && (t > t_0'), \\ y' &< -k'', \\ y &< -k''t + C', \\ &< k && (t > t_0''), \end{aligned}$$

* It is, of course, intended that p and q are positive integers with no common factor.

† For the sake of completeness we include steps in the argument which really duplicate former work; for example, Fowler, *loc. cit.*

‡ This theorem is part of a rather more general theorem whose proof has been given by the author, *loc. cit.* See p. 343, Table V.

contrary to hypothesis. Similarly, $y(t) < -k$ $(t > t_0)$ is also impossible.

We can now re-enunciate the remaining possibilities as follows :—

(*a*) For any positive k there exists a t_0 such that $y(t) > 0 (< 0)$ for $t > t_0$ and $y = y(t)$ cuts $y = k (-k)$ in a $\{t_s\}$.

(*b*) $y = y(t)$ oscillates finitely or infinitely about $y = 0$ cutting $y = \pm k$ and $y = 0$ in $\{t_s\}$'s.

(*c*) $y = y(t) \to 0$ as $t \to \infty$ with or without cutting $y = 0$ in a $\{t_s\}$.

Case (*a*) is impossible ; for let $y(t) > 0$ and let us start at t_0', a member of $\{t_s\}$, such that $t_0' > t_0$, $y_0 = y(t_0') = k$, $y_0' = y'(t_0') = -\delta$, $\delta > 0$. Then

$$\begin{aligned} y'' + y' &< 0 \qquad (t > t_0'), \\ y' + y &< y_0' + y_0 = k - \delta, \\ e^t y - e^{t_0'} k &< (k - \delta)(e^t - e^{t_0'}), \\ y &< k - \delta(1 - e^{(t_0' - t)}), \\ &< k, \end{aligned}$$

which contradicts the hypothesis.

Case (*b*) is also impossible. For consider any one of the assumed infinity of oscillations about $y = 0$. Let τ_s be the abscissa of the

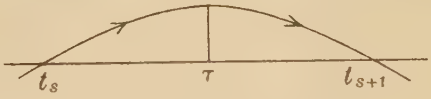

FIG. 1.

maximum of $y(t)$, which is easily proved to be the sole stationary value of $y(t)$ in the interval $t_s t_{s+1}$. On the upward arc $t_s \tau_s$, $y'(t) \geqslant 0$. Therefore

$$\begin{aligned} -y'' &\geqslant y^n, \\ -y''y' &\geqslant y^n y', \end{aligned}$$

$$\tfrac{1}{2}[y'(t_s)]^2 > \frac{1}{n+1}[y(\tau_s)]^{n+1}.$$

Similarly, from the downward arc $\tau_s t_{s+1}$,

$$\tfrac{1}{2}[y'(t_{s+1})]^2 < \frac{1}{n+1}[y(\tau_s)]^{n+1}.$$

Hence

$$|y'(t_s)| > \left(\frac{2}{n+1}\right)^{\frac{1}{2}} |y(\tau_s)|^{\frac{n+1}{2}} > |y'(t_{s+1})| > \left(\frac{2}{n+1}\right)^{\frac{1}{2}} |y(\tau_{s+1})|^{\frac{n+1}{2}} \quad \ldots \quad (13)$$

These inequalities state that both $|y'(t_s)|$ and $|y(\tau_s)|$ form decreasing monotonic sequences as $s \to \infty$. Hence either both $|y'(t_s)| \to 0$, $|y(\tau_s)| \to 0$ as $s \to \infty$, or else

$$|y'(t_s)| \to l, \qquad |y(\tau_s)| \to m, \qquad \left(l = \left(\frac{2}{n+1}\right)^{\frac{1}{2}} m^{\frac{n+1}{2}}\right).$$

Suppose now that $| y'(t_s) | \to l$, and consider the foregoing inequalities more exactly. We have on the upward arc $t_s\tau_s$

$$- y''y' = y^n y' + y'^2,$$

and therefore, in particular,

$$\tfrac{1}{2}[y'(t_s)]^2 = \frac{1}{n+1}[y(\tau_s)]^{n+1} + \int_0^{y(\tau_s)} | y' | \, dy.$$

Similarly, from the downward arc

$$\tfrac{1}{2}[y'(t_{s+1})]^2 = \frac{1}{n+1}[y(\tau_s)]^{n+1} - \int_0^{y(\tau_s)} | y' | \, dy.$$

Hence

$$\tfrac{1}{2}[y'(t_s)]^2 - \tfrac{1}{2}[y'(t_{s+1})]^2 > \int_0^{y(\tau_s)} | y' |_{\text{down. arc}} \, dy.$$

It is therefore necessary for consistency that

$$\int_0^{y(\tau_s)} | y' |_{\text{down. arc}} \, dy \to 0.$$

But as we trace y backwards from t_{s+1} *up* the downward arc $| y' |$ *increases* until $y^n = | y' |$, *i.e.* at least as far as $y^n = l$. Hence

$$\int_0^{y(\tau_s)} | y' |_{\text{down. arc}} \, dy > \int_0^{\sqrt[n]{l}} l dy = l^{\frac{n+1}{n}},$$

and the condition fails. It is therefore impossible that $l \neq 0$, and Case (*b*) itself is impossible. The proof of Theorem I. is completed.

The next stage depends upon an application of Hardy's results for equation (5*a*). Since t does not occur explicitly in equation (A) we can convert our equation into an equation of the first order in y and v, where $v = y'$, namely,

$$v\frac{dv}{dy} + v + y^n = 0 \quad . \quad . \quad . \quad . \quad (14)$$

Since by Theorem I. $y \to 0$ as $t \to \infty$ we are required to determine the asymptotic nature of the solutions of (14) as $y \to 0$, but it may be that $y = 0$, and therefore $y \to 0$ without $t \to \infty$. Equation (14) can be transformed by the substitution $y = 1/\eta$ into

$$- v\eta^2\frac{dv}{d\eta} + v + \frac{1}{\eta^n} = 0 \qquad (\eta \to \infty) \quad . \quad . \quad (15)$$

and then still further by the substitution $\eta = \zeta^q$ into an equation of form (5*a*). Thus to equation (15) and therefore to (14) we may apply Hardy's theorem that *there is an asymptotic relation of one of the forms*

$$a \succ b, \qquad a \sim - b, \qquad a \prec b \quad . \quad . \quad . \quad (16)$$

between any two terms a *and* b *as* $\eta \to \infty$ $(y \to 0)$.*

* The notation is that of Hardy, *Orders of Infinity*, Cambridge Tracts, No. 12. In exceptional cases, more than two terms being of equal highest order, we may have $a \sim ab$, a a constant (in Hardy's notation, $a \asymp b$) instead of $a \sim - b$.

We observe, further, that if any solution starts from given t_0 with finite values of y and y' it can only fail to be a *proper* solution by y or y' (y or v) becoming infinite as $t \to t_1$ ($> t_0$), and such an event can also be studied by means of Hardy's theorem. We shall thus prove Theorem II. :

Theorem II.—(1) *All solutions of* (A) *are proper solutions.* (2) *Solutions of* (A) *which never vanish for* $t > t_0$ *are such that either*

$$y = Ce^{-t} - \frac{\alpha C^n e^{-nt}}{n(n-1)} \qquad (\alpha \sim 1), \quad . \quad . \quad \text{[E-solutions,}$$

or

$$\pm y \sim \left[\frac{1}{(n-1)(t+C)}\right]^{\frac{1}{n-1}} \quad . \quad . \quad \text{[M-solutions,}$$

(1) Suppose $y \to \infty$ as $t \to t_1$. Then the only possibilities are

(*a*) $v \sim -y^n$, $v' \prec 1$. The latter implies $v \prec y$, which is a contradiction ; or

(*b*) $v' \sim -1$, $vv' \succ y^n$. The former implies $v \sim -y$, $vv' \sim -y$, which is a contradiction ; or

(*c*) $vv' \sim -y^n \succ v$. The former implies $\frac{1}{2}v^2 \sim -y^{n+1}/(n+1)$, which is consistent with the second relation, but is itself impossible owing to the essential difference of sign. There exist, therefore, no real solutions of (14) as $y \to \infty$.

Suppose, therefore, that $v = y' \to \infty$ as $y \to y_0$, $t \to t_1$. Then must $v' \sim -1$, $v \sim -y$, which contradicts $y \to y_0$. No solution, therefore, can cease to be a proper solution. This completes the proof of part (1).

(2) We have to determine possible forms of the proper solutions of equation (14), which are such that $y \to 0$, $t \to \infty$. By Hardy's theorem there are three possibilities.

Case (*a*). $vv' \sim -v \succ y^n$. This hypothesis yields

$$v \sim -y + C',$$

$$\frac{dy}{-y+C'} \sim dt,$$

which only allows $t \to \infty$ if $C' = 0$. In that case

$$y = Ce^{-t(1+\epsilon)} \qquad (\epsilon \to 0)$$

consistently with our original hypothesis. If greater exactitude is required we may put $v = -y + z$ and repeat the argument. We find (the details are left to the reader) that we must have

$$v = -y + \frac{\alpha y^n}{n} \qquad (\alpha \sim 1).$$

This equation is easily integrated approximately to give

$$y = Ce^{-t} - \frac{\alpha C^n e^{-nt}}{n(n-1)} \qquad (\alpha \sim 1) \quad . \quad . \quad (17)$$

We shall see directly that such solutions actually exist and are in fact the solutions constructed by Emden and used by Eddington. A convenient name for them will therefore be E-solutions.

Case (*b*). $-v \sim y^n \succ vv'$. This hypothesis yields

$$\left.\begin{aligned} -\frac{dy}{y^n} &\sim dt, \\ \frac{1}{y^{n-1}} &\sim (n-1)t + C, \\ \pm y &\sim \left[\frac{1}{(n-1)(t+C)}\right]^{\frac{1}{n-1}} \\ \pm \frac{dy}{dt} &\sim -\left[\frac{1}{(n-1)(t+C)}\right]^{\frac{n}{n-1}} \end{aligned}\right\}, \quad . \quad . \quad (18)$$

which are consistent with $v' \prec 1$. We shall prove later that such solutions actually exist. They are, of course, those rediscovered by Milne, and it seems proper to refer to them as M-solutions.

Case (*c*). $vv' \sim -y^n \succ v$. This case is impossible, for it implies

$$\tfrac{1}{2}v^2 \sim C - \frac{y^{n+1}}{n+1}.$$

$C = 0$ does not allow of a real solution. $C \neq 0$ implies that $v^2 \sim C$ in contradiction with $y^n \succ v$, $y \to 0$. The proof of Theorem II. is thus completed. We can now proceed to Part II. of the discussion, in which we make a new start and obtain the rest of the information that is desired.

§ 3. *Study of Equation* (A). *Part II. The Actual Forms of the Solutions and their Arrangement.*—Emden has studied and actually constructed numerically a solution of the (θ, ξ)-equation (2) for various values of n, in particular $n = 3$ when it is equivalent to equations (A). Such methods of numerical integration are usually equivalent to an existence theorem. We may therefore assume that Emden's typical solution exists for $n = 3$, leaving open for the present for what other values of n similar solutions of equations (A) exist. Since $n = 3$ is the astrophysically important equation, this procedure is the most convenient. For completeness we give the necessary existence theorems themselves for $n > 1$ in § 6. We discuss therefore now odd values of $n > 1$ which have an Emden solution similar to that for $n = 3$. This solution has the following properties :—

$$\theta = 1,\ \theta' = 0 \text{ at } \xi = 0,$$
$$[y = 0,\ y' = 1, \text{ at } \xi = 0,\ y \sim \xi = e^{-t}];$$

$\theta(\xi)$ first vanishes for a certain value $\xi = \xi_0$, and $\theta'(\xi) < 0$ everywhere in the range $0 < \xi \leqslant \xi_0$. It is one of the E-solutions of Theorem II.

On referring to the (y, t)-equation, we see that it does not contain t explicitly, and therefore if $y = y(t)$ represents Emden's solution,

$y = y(t + C)$ is another solution for any C. Hence, if $y = f(\xi)$ is Emden's solution,

$$\left.\begin{aligned} y &= f(C\xi) \qquad (C \text{ any constant}) \\ \theta &= \frac{1}{\xi} f(C\xi) \end{aligned}\right\} \quad . \quad . \quad (19)$$

is another solution. Any such solution is an E-solution of Theorem II. There exists therefore a family of E-solutions of the form (19), where f is the function constructed by Emden. In the (y, ξ)-plane they are of the simple half-sine-curve form shown in fig. 2. Every curve of the family is identical in type, only the ξ-scale changing from member to member. They have an envelope $y = m$, where m is the root of $f'(\xi) = 0$, a root easily shown to be the unique stationary value of $f(\xi)$ for the range $0 \leqslant \xi \leqslant \xi_0$, in which alone we are interested. Their various intersections must lie in the relative positions shown, and there

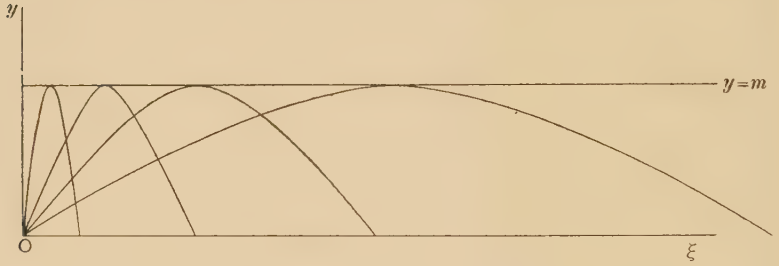

FIG. 2.

can be no others. Any one curve cuts in order all curves for greater C before it touches the envelope itself and after they have touched it. It cuts in order all curves of lesser C after it touches the envelope itself and before they touch it.*

Multiplication by $1/\xi$ to form the (θ, ξ)-curves distorts the picture, but cannot affect contacts or relative arrangements of intersections. It therefore yields the curves shown in fig. 3, no other arrangements or intersections being possible. The envelope is the rectangular hyperbola

$$\theta = m/\xi \quad . \quad . \quad . \quad . \quad (20)$$

The remaining part of the argument is most easily carried through in the (θ, ξ) diagram.

There are in fact no other E-solutions except this family. This is guaranteed by the uniqueness theorem for the solution of a differential equation (see § 6 for details), for our family includes all initial conditions of type $\theta(0) = C, \theta'(0) = 0$, which are characteristic of any E-solution. We need, however, to appeal to this theorem with care as it has here to be applied to a singular point $\xi = 0$. It is then sufficient to group the

* All the facts are intuitively obvious from the geometry in this simple case. In the more complicated case of § 9 a more detailed discussion is given which covers this case too.

solutions into this family of E-solutions and all other solutions, and proceed to discuss *what all the others can do*, remembering that they cannot be E-solutions.

With this object in view let us start from any initial point P ($\xi = \xi_1$, $\theta \geqslant 0$) and follow any solution through P inwards towards $\xi = 0$. It is convenient in this section to reckon slopes as positive when θ increases going towards the origin. The principle of the argumentation

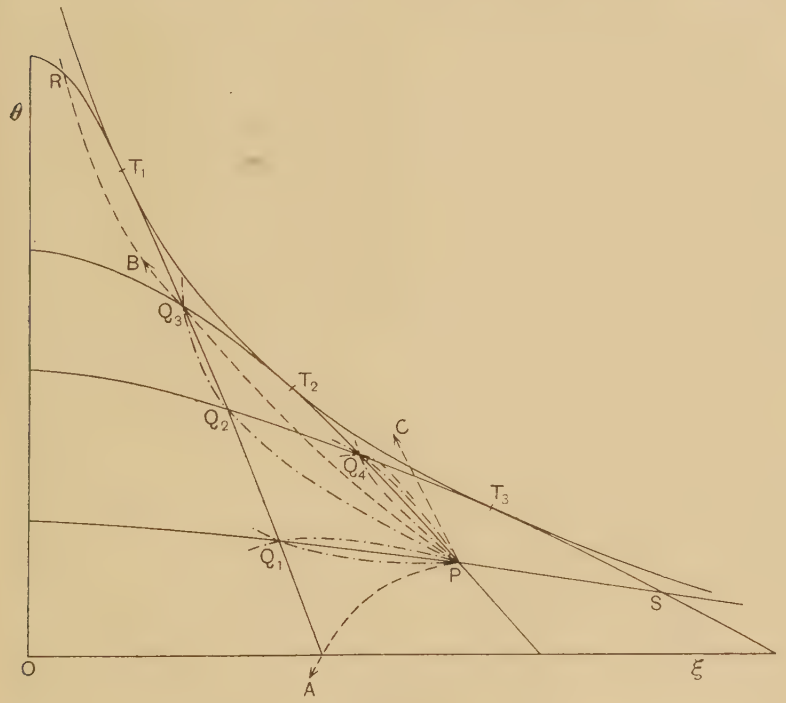

Fig. 3.

is that *by the uniqueness theorem no other solution may ever touch any member of the family of E-solutions, for if it does it must be an E-solution itself.*

Consider first any position of the point P below the envelope. Through such a P there pass two, and only two, E-solutions of slopes α and β, $\alpha > \beta$, both *positive* in the present sense. The corresponding E-solutions themselves may be referred to as E_α and E_β respectively —if necessary as $E_\alpha(P)$, $E_\beta(P)$. In the limiting case, when P is on the envelope, E_α and E_β coincide and $\alpha = \beta$. In the limiting case when P is on $\theta = 0$, $\beta = 0$, and E_β is $\theta = 0$. In fact, $\theta = 0$ is the limiting form of the E-solutions as $C \to 0$ and $\xi = 0$ their limit as $C \to +\infty$. The detailed behaviour of *any* solution through P can be deduced quite simply from the results of Part I. and the uniqueness theorem.

Case (1), $\theta' < \beta$. There are three *a priori* possibilities :

(i) $\theta = \theta(\xi)$ cuts E_β again at ξ_2, (Q_1), and $\theta(\xi) > 0$, $\xi_2 \leqslant \xi \leqslant \xi_1$.
(ii) $\theta = \theta(\xi)$ remains between $\theta = E_\beta(\xi)$ and $\theta = 0$ as $\xi \to 0$.
(iii) $\theta(\xi_2) = 0$, $0 < \xi_2 < \xi_1$.

(i) This is impossible by the uniqueness theorem. For consider the continuum of E-solutions for which $0 \leqslant C \leqslant C_\beta(P)$ (with an obvious notation). As C diminishes, their intersections with the two fixed curves $E_\alpha(P)$ and $E_\alpha(Q_1)$ move steadily downwards towards the line $\theta = 0$, which is the limit of the E-solutions as $C \to 0$. It is geometrically obvious that one of these E-solutions must touch the proposed solution, and this is impossible. The argument can be given rigorously, as follows. If for $C = C'$ the E-solution has points in common with $\theta = \theta_1(\xi)$, then it also has points in common with it for $C > C'$. If for $C = C'$ the E-solution has no points in common with $\theta = \theta(\xi)$ then it still has no points in common for $C < C'$. For a sufficiently small C the second type of E-solution exists. We can therefore make a Dedekind section in the C's by the criterion of having or not having common points, and the section determines a C_0 for which the corresponding E-solution can only touch $\theta = \theta(\xi)$.

(ii) This is impossible, because by Theorem I. the solution must then be either an M-solution or an E-solution. It cannot be an M-solution because it remains below a given E-solution. It cannot be an E-solution since it is not (near P) a member of the family of E-solutions.

We conclude that if $\theta'(\xi_1) < \beta$ at P, $\theta = \theta(\xi)$ certainly cuts $\theta = 0$ at ξ_2, $0 < \xi_2 < \xi_1$.

Case (2), $\alpha > \theta' > \beta$. There are five *a priori* possibilities, as follows. The lettering of points refers to fig. 3.

(i) $\theta = \theta(\xi)$ remains between E_α and E_β as $\xi \to 0$.
(ii) $\theta = \theta(\xi)$ first cuts E_β at ξ_2, Q_1, remaining below E_α for $\xi_2 \leqslant \xi < \xi_1$.
(iii) $\theta = \theta(\xi)$ first cuts E_α at ξ_2, Q_4, between P and T_2 where E_α touches the envelope, or at T_2, remaining above E_β for $\xi_2 \leqslant \xi < \xi_1$.
(iv) $\theta = \theta(\xi)$ first cuts E_α at ξ_2, Q_3, beyond T_2 at such a slope that $\theta'(\xi_2) > \alpha(Q_3)$.
(v) $\theta = \theta(\xi)$ first cuts E_α at ξ_2, Q_3, beyond T_2 at such a slope that $\alpha(Q_3) > \theta'(\xi_2) > \beta(Q_3)$.

(i) is obviously impossible by the argument already used in Case (1) (ii) above.

(ii) is impossible by the uniqueness theorem. For if we take the E-solution PQ_1 and increase its C, its intersections with the fixed E-solutions Q_1Q_2 and PQ_4 move up through Q_2Q_4 to a limiting position Q_3T_2 as C approaches $C_\alpha(P)$. Thus we must reach an E-solution which has no points in common with $\theta = \theta(\xi)$ between $Q_1Q_2Q_3$, and PQ_4T_2, and therefore, by a Dedekind section, one that touches $\theta = \theta(\xi)$.

(iii) is impossible by the same argument. We keep the E-solutions

PQ_1, Q_4Q_2 fixed and move their intersections with PQ_4 to the left by increasing the C of this E-solution. Since PQ_4 then has $\xi = 0$ as a limit, the E-solution must clear $\theta = \theta(\xi)$ and contact must occur.

(iv) is also impossible by the same argument. For if $\theta = \theta(\xi)$ is to cut $E_\alpha(P)$ ($\equiv E_\beta(Q_3)$) again at Q_3 at a slope greater than $\alpha(Q_3)$ it must have previously cut $E_\alpha(Q_3)$ in order to get below it, at Q_2 say. Applying the same argument to the arc Q_2Q_3 we reach a contact and the case fails.

We conclude that *if* $\beta < \theta'(\xi_1) < \alpha$ *at* P *then* $\theta = \theta(\xi)$ *remains above* E_β *and below* E_α *until it cuts the latter again at a point* Q_3 *beyond the contact of* E_α *with the envelope, and in such a way that* $\beta(Q_3) < \theta'(\xi_2) < \alpha(Q_3)$.

We can, however, say much more than this, for the argument can be repeated for Q_3 as for P, leading to a new similar intersection at R, say, and so on indefinitely. The solution $\theta = \theta(\xi)$, starting in this way at P, as $\xi \to 0$, must ultimately pass above every E-solution and lie below the envelope; we conclude that $\theta = \theta(\xi) \to +\infty$ *and is an M-solution.*

Case (3), $\theta' > \alpha$. This case is rather more complicated and it is well to dispose of the easier alternatives first.

(i) $\theta = \theta(\xi)$ passes above the envelope. The following argument applies also to any solution started from a point P above the envelope. Such a solution cannot remain above the envelope, for then y ($= \xi\theta$) does not tend to zero. It must therefore recross the envelope from above downwards (without touching it and therefore an E-solution). Since all slopes are continuous and on the envelope $\alpha = \beta =$ slope of envelope, one can find a point P on the solution near and below the envelope where we have the inequalities of Case (1). There is, therefore, for any such solution a point $\xi_2 < \xi_1$ at which $\theta(\xi_2) = 0$.

(ii) $\theta = \theta(\xi)$ must not touch the envelope.

(iii) $\theta = \theta(\xi)$ remains below the envelope. In this case it must cut $E_\alpha(P)$ again before T_2, at Q_4, $\xi = \xi_2$, say. It may cut it so that $\theta'(\xi_2) < \beta(Q_4)$. In that case we are back again in Case (1) and $\theta(\xi_2') = 0$, $\xi_2' < \xi_1$. It may finally cut $E_\alpha(P)$ again at Q_4, so that

$$\alpha(Q_4) > \theta'(\xi_2) > \beta(Q_4).$$

If it does this, as shown in fig. 3, we have a contradiction with the uniqueness theorem. For keeping Q_4T_3, PS fixed let the E-solution Q_4P move to the right by diminishing C towards its limiting position T_3S. Unless $\theta = \theta(\xi)$ also intersects T_3S we form a Dedekind section and have a contradiction with the uniqueness theorem. We are left, therefore, with the curious alternative shown more clearly in fig. 4. If there were more than one such arc $Q_4'P'$ we could make a definite choice of that one nearest to T_3.

Consider more closely the arc $P'Q_4'$. The relationship of $\theta'(\xi)$ to α and β at P' is the same as at P. It must also be the same at Q_4' as at Q_4 or we should meet obvious contradictions. We therefore can repeat the whole argument and find a contradiction with the uniqueness theorem, unless an internal part of the arc $P'Q_4'$ of $\theta = \theta(\xi)$ lies to the right of $T_3'S'$. The argument can be repeated indefinitely, unless we

meet with a contradiction with the uniqueness theorem after a finite number of steps. At each step the points $P^{(s)}Q_4^{(s)}$ move monotonically towards each other along the given solution $\theta = \theta(\xi)$. They cannot have non-coincident limits $P^{(l)}Q_4^{(l)}$, for we could always repeat the argument for the limiting arc itself. We should therefore necessarily have $P^{(l)} \equiv Q_4^{(l)}$ at which point $\theta = \theta(\xi)$ would touch an E-solution and contradict the uniqueness theorem. If, therefore, $\theta'(\xi_1) > \alpha(P)$

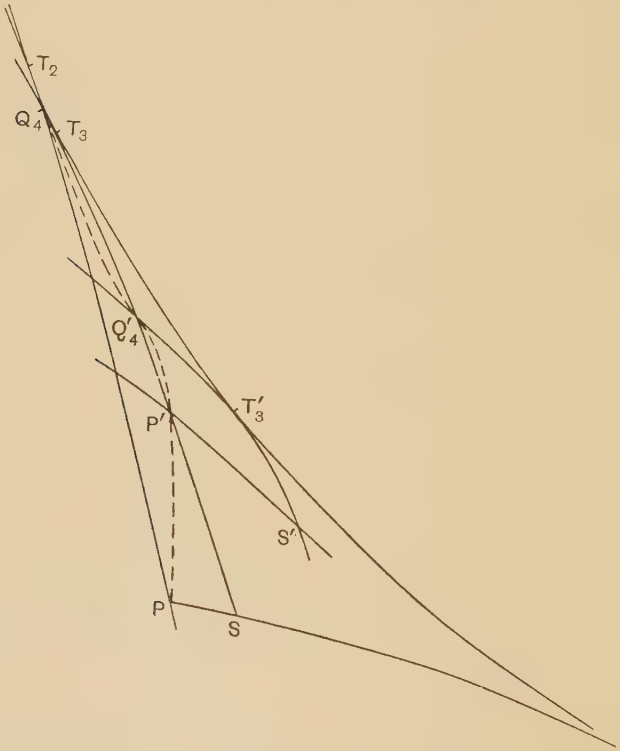

Fig. 4.

and $\theta = \theta(\xi)$ remains below the envelope, it is impossible that $\alpha(Q_4) > \theta'(\xi_2) > \beta(Q_4)$. We conclude that *if* $\theta'(\xi_1) > \alpha(P)$ *there exists* $a\ \xi_2' < \xi_1$ *such that* $\theta(\xi_2') = 0$.*

We collect the results of this section in Theorem III.

Theorem III.—Through any point P, ξ_1, *below the envelope,* $\theta = m/\xi$, *of the E-solutions of equation* (A) *pass two E-solutions with slopes* α, β, $\alpha > \beta$. (1) *If* $\alpha > \theta'(\xi_1) > \beta, \theta = \theta(\xi)$ *is an M-solution and* $\theta(\xi) \to +\infty$ *as* $\xi \to 0$. (2) *If* $\theta'(\xi_1) = \alpha$ *or* $\theta'(\xi_1) = \beta$ *the solution is an E-solution.*

* It seems to be possible to prove by a similar argument the more exact result that, *if* $\theta'(\xi_1) > \alpha(P)$, $\theta = \theta(\xi)$ *rises above the envelope.* This result, however, would not be of any great value to us here.

(3) *If* $\theta'(\xi_1) > \alpha$ *or* $\theta'(\xi_1) < \beta$ *the solution cuts* $\theta = 0$ *again at a point* ξ_2 *such that* $0 < \xi_2 < \xi_1$.

Any solution through a point P, ξ_1, *above or on the envelope cuts* $\theta = 0$ *again at* ξ_2, *where* $0 < \xi_2 < \xi_1$, *except that through a point* P *on the envelope there passes the single E-solution which touches the envelope there.*

This theorem is the part of our work which is of chief importance to astrophysicists.

§ 4. *Study of Equation* (A). *Part III. Non-existence of Oscillating Solutions as* $\xi \to 0$, $t \to \infty$.—We can now use the results of the last section to complete our study. We proved in Part I. that if $y = y(t)$ continually oscillates about $y = 0$ as $t \to \infty$, then $| y'(t) | \to 0$. Therefore as $\xi \to 0$

$$| y'(\xi_s) | = o\left(\frac{1}{\xi}\right), \qquad | \theta'(\xi_s) | = o\left(\frac{1}{\xi^2}\right).$$

But, at the zero point of an E-solution $\theta = \theta_E(\xi)$,

$$f(C\xi) = 0, \qquad C\xi = q;$$

$$\theta_E'(\xi) = \frac{d}{d\xi}\left\{\frac{1}{\xi}f(C\xi)\right\} = \frac{C}{\xi}f'(q) = \frac{q}{\xi^2}f'(q).$$

Hence if the solution continually recrosses the axis $\theta = 0$ as $\xi \to 0$ we must arrive at a zero, while still $\xi \geqslant \epsilon > 0$, at which

$$| \theta'(\xi) | \leqslant | \theta_E'(\xi) |.$$

But this must be the last zero; for the conditions of Theorem III. (1) or (2) are now satisfied and the solution now in general tends to $\pm \infty$ as an M-solution or, in the particular case of equality to $\pm C$, as an E-solution. Though we have argued all the time about positive values of y, all arguments transfer at once to negative values so long as $(-y)^n = -(y)^n$.

As a study in the form near infinity of the proper solutions of our differential equations (A) we can combine our results into the following theorem:—

Theorem IV.—As $x, t \to \infty$ *no proper solution of the equations* (A)

$$\theta'' + x^{-n-1}\theta^n = 0, \qquad y'' + y' + y^n = 0,$$

where $n = p/q > 1$, *and* p, q *are odd, can oscillate about zero. In terms of* (y, t), *every solution* $y = y(t) \to 0$ *as* $t \to \infty$ *in one or other of the asymptotic forms*

$$y = Ce^{-t} - \frac{aC^n e^{-nt}}{n(n-1)} \qquad (a \sim 1),$$

$$\pm y \sim \left[\frac{1}{(n-1)(t+C)}\right]^{\frac{1}{n-1}}.$$

The first form is exceptional.

The proof of this theorem is so far completed for $n = 3$ and for any other values of n to which Emden's construction applies. We shall complete the justification of the theorem for the full range $n > 1$

by proving *ad hoc* an existence and uniqueness theorem for the family of E-solutions, in § 6.

§ 5. *Study of Equations* (A) *when* $n = p/q > 1$ *but* p *is even and* q *is odd, or* q *is even.* The study of the first case, characterized by

$$(-y)^n = y^n$$

is somewhat simpler than the case whose discussion has just been completed. For positive y there is no essential difference. Every argument which we have used in Theorems I.–IV. remains unaffected so far as it applies to the region $y > 0$. It is only when $y = y(t)$ crosses the axis of y that any difference can enter. When, however, this happens we see on putting $y = -\eta$ that we have then the form

$$\eta'' + \eta' - \eta^n = 0 \qquad \eta > 0 \quad . \quad . \quad . \quad (21)$$

Let us consider what happens to any solution $\eta = \eta(t)$ of equation (21), which starts at $t = t_0$, say, with the initial conditions

$$\eta(t_0) = 0, \qquad \eta'(t_0) > 0.$$

It is easy to see that $\eta'(t) > 0$ for $t > t_0$. For

$$\eta'' + \eta' > 0,$$

$$\frac{d}{dt}(e^t\eta') > 0,$$

$$\eta' > e^{t_0 - t}\eta'(t_0) > 0 \qquad (t > t_0).$$

Thus either $\eta(t) \to l$ or $\eta(t) \to +\infty$. But if $\eta(t) \to l$, it follows that

$$\frac{d}{dt}(e^t\eta') > e^t(l^n - \epsilon) \qquad (t > t_0),$$

$$\eta' > \kappa > 0 \qquad (t > t_0'),$$

which contradicts $\eta(t) \to l$. Therefore $\eta(t) \to \infty$. The solution must therefore cease to be a proper solution unless also $t \to \infty$. But we can now consider

$$v\frac{dv}{d\eta} + v - \eta^n = 0 \qquad (\eta \to +\infty),$$

using Hardy's theorem. There are three cases:

(i) $v \sim \eta^n \succ v\dfrac{dv}{d\eta}$. This implies

$$\frac{d\eta}{\eta^n} \sim dt,$$

$$-\frac{1}{n-1}\left(\frac{1}{\eta^{n-1}} - \frac{1}{\eta_0^{n-1}}\right) \sim t - t_0,$$

$$t \to t_1.$$

(ii) $\dfrac{dv}{d\eta} \sim -1$, $v \succ \eta^n$. But the first relation implies $v \sim -\eta$ which

is inconsistent with $v \succ y^n$ when $n > 1$. This case is therefore impossible.

(iii) $v\dfrac{dv}{d\eta} \sim \eta^n \succ v$. This implies $\frac{1}{2}v^2 \sim \dfrac{1}{n+1}\eta^{n+1}$ and is self-consistent when $n > 1$. By the same argument, as in Case (i), we are led to $t \to t_1$. It is not possible that $t \to \infty$. As soon as the solutions of our present equation (A) cross over to negative y they must sooner or later, but for finite x or t, cease to be proper solutions.

The study of the second case, in which no real value of $(-y)^n$ can exist, is still simpler. For if $y(t) \to 0$ as $t \to t_1$ the solution must there cease to be a proper solution, unless perhaps $y'(t) \to 0$ also. But in that case the solution would touch the solution $y = 0$, which is impossible. We can therefore now enunciate the following theorem :—

Theorem V.—The equations

$$\theta'' + x^{-n-1}\theta^n = 0, \qquad y'' + y' + y^n = 0,$$

where $n = p/q > 1$ *and one of* p, q *is even, cannot have a proper solution as* $x, t \to \infty$ *which cuts the axis* $y = 0$ *from the side of positive* y, *and* a fortiori *cannot have a solution oscillating about* $y = 0$. *For positive* y *proper solutions exist, which must tend to zero monotonically, must be arranged in the same way, and have the same asymptotic forms as the corresponding solutions described in Theorems I.–IV.*

§ 6. *Existence and Uniqueness Theorem for E-solutions. Completion of the Proofs of Theorems I.–V.*—The existence and uniqueness of the solution of equation (A), which starts from any non-singular point with given values of y and y' or θ and θ', are guaranteed by a theorem sufficiently familiar about which no more need be said.* Since, however, the whole argument is based on the existence of the E-solutions (and to a less degree on their uniqueness) and these are defined by their properties at a singular point $\xi = 0$, it may be convenient to give here an *ad hoc* proof of their existence and uniqueness. The simple proof here given is of a standard form and makes no claim to novelty.†

Take the equation in the form

$$v' = -1 - y^n/v \quad . \quad . \quad . \quad . \quad (22)$$

and attempt to construct a solution, for which $v \sim -y$, $y \to 0$, by Picard's iterative process with

$$\begin{aligned} v_1 &= -y, \\ v_2' &= -1 - y^n/v_1 \qquad (v_2(0) = 0), \\ &\cdots \\ v_s' &= -1 - y^n/v_{s-1} \qquad (v_s(0) = 0). \\ &\cdots \end{aligned}$$

Suppose we have proved that for $0 \leqslant y \leqslant y_0$

$$Ay \leqslant -v_{s-1} \leqslant y \qquad (0 < A < 1) \quad . \quad . \quad . \quad (23)$$

* See, for example, Picard, *Cours d'Analyse*, vol. ii. chap. xi.

† A convenient reference for existence and uniqueness theorems at singular points is Ince, *Ordinary Differential Equations*, chap. xii.

Then

$$-v_s' < 1, \quad -v_s < y,$$

and

$$-v_s' > 1 - \frac{1}{A}y^{n-1}, \qquad -v_s > y - \frac{1}{nA}y^n.$$

We can therefore choose A and y_0, for example, by taking

$$y_0^{n-1} < nA(1 - A),$$

so that

$$Ay \leqslant -v_s \leqslant y \qquad (0 < A < 1).$$

Therefore, since relation (23) is satisfied by v_1, it is satisfied by v_s for all s.

It can next be shown that $\underset{s\to\infty}{\mathrm{Lt}}\, v_s$ exists and is the function v we require. For

$$\underset{s\to\infty}{\mathrm{Lt}}\, v_s = v_1 + \sum_{s=1}^{\infty}(v_{s+1} - v_s) \quad . \quad . \quad . \quad (24)$$

and

$$v_{s+1}' - v_s' = \frac{y^n}{v_s v_{s-1}}(v_s - v_{s-1}),$$

$$|\, v_{s+1}' - v_s' \,| < \frac{y^{n-2}}{A^2}\,|\, v_s - v_{s-1} \,| \qquad (0 \leqslant y \leqslant y_0).$$

If therefore

$$|\, v_s - v_{s-1} \,| < k_{s-1} \qquad (0 \leqslant y \leqslant y_0),$$

$$|\, v_{s+1} - v_s \,| < k_{s-1} \cdot \frac{1}{A^2}\int_0^{y_0} y^{n-2}dy = k_{s-1} \cdot \frac{y_0^{n-1}}{(n-1)A^2},$$

$$< \epsilon \,.\, k_{s-1} \qquad (\epsilon < 1),$$

provided y_0 be chosen sufficiently small, since $n > 1$. Thus

$$|\, v_{s+1} - v_s \,| < \epsilon^s k_0,$$

which guarantees that the series (24) converges absolutely and uniformly, and that a function $v(y)$ exists such that

$$\underset{s\to\infty}{\mathrm{Lt}}\, v_s = v(y)$$

uniformly in the range $0 \leqslant y \leqslant y_0$. Further, since in (23) A can be chosen as near 1 as we please by suitable choice of y_0, we have also

$$v(y) \sim -y.$$

Further, $v = v(y)$ satisfies the differential equation, for

$$v(y) = -y - \underset{s\to\infty}{\mathrm{Lt}} \int_0^y y^n dy / v_s(y),$$

$$= -y - \int_0^y \frac{y^n dy}{v(y)} - \underset{s\to\infty}{\mathrm{Lt}} \int_0^y \frac{y^n dy}{v_s v(y)}\{v(y) - v_s\}.$$

But for $0 \leqslant y \leqslant y_0$, v_s, and therefore $v(y)$, satisfies (23), and since the convergence of (24) is uniform $| v(y) - v_s | < \epsilon$ for sufficiently large s over the whole of the same interval. The limit of the last term is therefore zero, and

$$v(y) = -y - \int_0^y \frac{y^n dy}{v(y)},$$

which is what we required to prove.

Finally, we can show that if $v^*(y)$ satisfies the conditions $v^*(y) \to 0$ $(y \to 0)$, $| v^* | > Ay$ $(0 \leqslant y \leqslant y_0)$, and is also a solution of (22), then $v^*(y) \equiv v(y)$, so that $v(y)$ is unique. For let Δ be the numerical value of the difference $v^*(y) - v(y)$. Then

$$\frac{d\Delta}{dy} = \frac{y^n}{v^*(y)v(y)}\Delta \qquad (0 \leqslant y \leqslant y_0),$$

$$< Ky^{n-2}\Delta \qquad (0 \leqslant y \leqslant y_0),$$

where K is a definite constant. Thus

$$\log \frac{\Delta(y)}{\Delta(y_1)} < \frac{K}{n-1}(y^{n-1} - y_1^{n-1}) \qquad (0 \leqslant y, y_1 \leqslant y_0).$$

Now let $y_1 \to 0$. Then $\Delta(y_1) \to 0$, but the right-hand side remains bounded since $n > 1$. Therefore $\Delta(y) = 0$, and the solutions are identical. We combine these results into the following lemma. Once it is away from the critical point the solution can be continued uniquely by the usual methods.

Lemma I.—There exists a solution $v = v(y)$ *of the equation*

$$vv' + v + y^n = 0 \qquad (n > 1),$$

which satisfies $v \sim -y$ *as* $y \to 0$, *and this solution is unique.*

The unique $v = v(y)$ leads of course to the corpus of E-solutions. It is well to trace through this last step. We have

$$v = \frac{dy}{dt} = -\xi\frac{dy}{d\xi} = v(y) = -y + \frac{\alpha y^n}{n} \quad . \quad . \quad (25)$$

where $\alpha \sim 1$ as $y \to 0$. Therefore

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\xi} = \int_{y_0}^{y} \frac{dy}{-v(y)},$$

or, say,

$$\log \frac{\xi}{\xi_0} = g(y) - g(y_0),$$

where, as $y_0 \to 0$,

$$g(y_0) \sim \log y_0 + o(1).$$

There must exist a certain range, $0 \leqslant y \leqslant y_1$, in which $g(y)$ is a definite one-valued monotonic function of y. Now let $y_0 \to 0$. Then must also $\xi_0 \to 0$ and, in order that there may be any solution of equation (25), in such a way that

$$y_0/\xi_0 \to C. \quad . \quad . \quad . \quad . \quad (26)$$

In that case $$\log C\xi = g(y).$$

Since $g(y)$ is a continuous monotonic function of y in the range $0 \leqslant y \leqslant y_1$ we can take exponentials and invert this functional relation, obtaining a unique relation

$$y = f(C\xi) \quad . \quad . \quad . \quad . \quad (27)$$

such that

$$y/\xi \to C, \qquad (\xi \to 0), \qquad (n > 1) \quad . \quad . \quad (28)$$

which is the family of E-solutions previously assumed.

It does not follow, of course, from what we have already proved that for all values of $n > 1$ the family (27) consists of curves which have a zero for $C\xi = X > 0$, as shown in fig. 2, and therefore with all the essential properties of Emden's curves for $n = 3$. This, however, follows at once for p, q odd from the fact that as $t \to \infty$ all the solutions of $y'' - y' + y^n = 0$ must oscillate about $y = 0$,* and therefore all the solutions of $y'' + y' + y^n = 0$ as $t \to -\infty$, and therefore $y = y(\xi)$ as $\xi \to +\infty$. For other values of n we have to consider the equation $y'' - y' + y^n = 0$, known not to have oscillating solutions, and ask further whether it can have a monotonic solution for which $y > 0$ as $t \to +\infty$. The various possibilities can all be handled with the help of Hardy's theorem applied to the equation

$$\frac{dv}{dy} = +1 - \frac{y^n}{v},$$

and it is easily found that there is no possibility of such solutions. Monotonic proper solutions exist only for $y < 0$ as $t \to +\infty$. The E-solutions, therefore, which start positive near $\xi = 0$ must either become negative or cease to be proper solutions—in either case by reaching a point ξ_0 at which $y(\xi_0) = 0$. From the (y, ξ) form of equation (A) y' must continually decrease from $\xi = 0$ to $\xi = \xi_0$, so that the simple geometrical form of fig. 2 of the family (27) is assured.

From the family (27) and condition (28) we can at once deduce the existence and nature of the θ, ξ curves. It is to be observed, however, that though they all possess the property

$$\theta(\xi) \to C \qquad (\xi \to 0)$$

they do not all possess the property

$$\theta'(\xi) \to 0 \qquad (\xi \to 0),$$

which, in fact, they possess only when $n > 2$, as may be seen from their asymptotic form (17). We find that

$$\theta'(\xi) \to -\tfrac{1}{2}C^2 \qquad (n = 2),$$

$$\theta'(\xi) \to -\infty \qquad (1 < n < 2).$$

In this case there is a *cusp* at $\xi = 0$ with a vertical tangent. The figures 3 and 4 therefore no longer properly represent the θ, ξ curves.

* Fowler, *loc. cit.*, p. 344, Table VI.

The curves for $n \leqslant 2$ are, however, entirely unaltered in their spatial relationships between themselves and their envelope, and the discussion we have given in § 3 holds unaltered.

The proofs of all our theorems are thus completed for all values of $n > 1$, for which alone they are true.

§ 7. *Study of the Equations* (n = p/q > 1, p, q *odd*)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \xi^{-4-\sigma}\theta^n = 0, \quad y'' + \xi^{-\sigma-n-3}y^n = 0, \quad \theta'' + x^\sigma\theta^n = 0$$
$$(\sigma + n + 1 < 0) \quad . \quad . \quad . \quad . \quad . \quad (B)$$

Part I. Preliminary Remarks.—This equation includes the equation used by Milne for gas spheres obeying Fermi-Dirac statistics. Much of the argument is closely the same for equation (B) as for equation (A) and need not be given in such detail.

We start by recalling what is already known about the asymptotic form of solutions of the equation

$$\theta'' + x^\sigma\theta^n = 0 \qquad (\sigma + n + 1 < 0, \qquad n > 1) \qquad . \quad (28')$$

It has been shown * that *as* $x \to \infty$ *all proper solutions of equation* (28′) *which are ultimately monotonic are of one or other of the forms*

$$\text{(i)} \pm\theta = x^\epsilon, \qquad \text{(ii)} \pm\theta = x^{1+\epsilon}, \qquad \text{(iii)} \pm\theta = x^{-\frac{\sigma+2}{n-1}+\epsilon},$$

where $\epsilon \to 0$. It is of course always implied that such ϵ's are real. This result was obtained by a study of the conditions under which $x^\sigma\theta$ may or may not be monotonic for every value of σ, and reference should be made to the paper cited for the proof. It has not been proved whether or not oscillating solutions can exist. The possible forms of y for a monotonic positive θ are therefore, as $\xi \to 0$,

$$\text{(i)}\ y = \xi^{1+\epsilon}, \quad \text{(ii)}\ y = \xi^\epsilon, \quad \text{(iii)}\ y = \xi^{\frac{\sigma+n+1}{n-1}+\epsilon} \qquad . \quad (29)$$

For the last of these $y \to \infty$.

We can obtain at once additional information bearing on possible oscillating solutions. Consider a loop on such a solution, as shown in fig. 1, the variables now being (θ, x), the zeros x_s, the maxima ξ_s. On the upward arc, since $\sigma < 0$,

$$-\theta'' > \xi_s{}^\sigma\theta^n,$$
$$-\theta''\theta' > \xi_s{}^\sigma\theta^n\theta',$$
$$\tfrac{1}{2}[\theta'(x_s)]^2 > \frac{1}{n+1}\xi_s{}^\sigma[\theta(\xi_s)]^{n+1}.$$

Similarly, from the downward arc

$$\tfrac{1}{2}[\theta'(x_{s+1})]^2 < \frac{1}{n+1}\xi_s{}^\sigma[\theta(\xi_s)]^{n+1}.$$

* See Fowler, *loc. cit.*, pp. 324, 325, Tables II., III.

Hence the terms $|\theta'(x_s)|$ form a decreasing monotonic sequence as $s \to \infty$.

Let us now trace such a solution more closely as it leaves $x = x_s$. Since $\theta'' < 0$ until $x = x_{s+1}$ we have

$$\theta'(x) \leqslant \theta'(x_s), \qquad \theta(x) \leqslant \theta'(x_s)(x - x_s) \qquad (x_s < x \leqslant x_{s+1}).$$

Therefore in this range

$$-\theta'' = x^\sigma \theta^n \leqslant x^\sigma (x - x_s)^n [\theta'(x_s)]^n.$$

Integrating this inequality,

$$\theta'(x_s) - \theta'(x) \leqslant [\theta'(x_s)]^n \int_{x_s}^{x} x^\sigma (x - x_s)^n dx.$$

It follows that $\theta'(x)$ cannot become negative until

$$\theta'(x_s) < [\theta'(x_s)]^n \int_{x_s}^{x} x^\sigma (x - x_s)^n dx\,;$$

that is never, so that the solution is after all monotonic, if

$$1 > [\theta'(x_s)]^{n-1} \int_{x_s}^{\infty} x^\sigma (x - x_s)^n dx,$$

$$= [\theta'(x_s)]^{n-1} x_s^{\sigma+n+1} \mathrm{M},$$

where M is a number depending only on σ and n. Since on a would-be oscillating solution $\theta'(x_s)$ diminishes as $s \to \infty$ and $x_s^{\sigma+n+1} \to 0$, this condition must eventually be satisfied and the solution must be ultimately monotonic, of one or other of the forms specified.

We can next show that monotonic solutions of type (iii), $\pm\theta = x^{-\frac{\sigma+2}{n-1}+\epsilon}$ $(\epsilon \to 0)$, are impossible. The argument is the same for either sign when p, q are odd. Suppose, then, that

$$\theta = x^{-\frac{\sigma+2}{n-1}+\epsilon};$$

then

$$-\theta'' = x^{-2-\frac{\sigma+2}{n-1}+\epsilon'} \qquad (\epsilon' \to 0),$$

and, since $\sigma + n + 1 < 0$,

$$-2 - \frac{\sigma+2}{n-1} > -1.$$

The last relation can therefore be integrated and takes the form

$$-\theta' = x^{-1-\frac{\sigma+2}{n-1}+\epsilon''} \qquad (\epsilon'' \to 0),$$

which can be integrated again, giving

$$-\theta = x^{-\frac{\sigma+2}{n-1}+\epsilon'''} \qquad (\epsilon''' \to 0).$$

On account of the sign this is necessarily inconsistent with the proposed form, which is therefore impossible.

A similar argument obviously shows that the relation

$$\theta \geqslant \lambda x^{-\frac{\sigma+2}{n-1}} \qquad (x > x_0),$$

where λ is any positive constant, is impossible for any solution.

We can now assert therefore that all proper solutions are ultimately monotonic and of type (i) or (ii). We can further refine these asymptotic forms.

Type (i), $\theta = x^{\epsilon}$ $\quad (\epsilon \to 0)$. It follows that

$$\begin{aligned} -\theta'' &= x^{\sigma+\epsilon'} \qquad (\sigma < -2), \\ -\theta' &= \mathrm{D} - x^{\sigma+1+\epsilon''}, \\ -\theta &= \mathrm{D}x - \mathrm{C} + x^{\sigma+2+\epsilon'''}. \end{aligned}$$

This is inconsistent with our hypothesis unless $\mathrm{D} = 0$, when it takes a form which we may write as

$$\theta = \mathrm{C} - x^{\sigma+2+\epsilon}.$$

This can be made still more exact by repetition of the argument. For instance, we have by one more step

$$\theta = \mathrm{C} - \frac{a\mathrm{C}^n x^{\sigma+2}}{(\sigma+1)(\sigma+2)} \qquad (a \sim 1) \quad . \quad . \quad (30)$$

This type is of course the family of E-solutions whose existence we shall establish rigorously in § 8. In terms of (y, ξ) this asymptotic form is

$$y = \mathrm{C}\xi\left[1 - a\frac{\mathrm{C}^{n-1}\xi^{-(\sigma+2)}}{(\sigma+1)(\sigma+2)}\right] \qquad (a \sim 1) \quad . \quad . \quad (31)$$

Type (ii), $\theta = x^{1+\epsilon}$ $\quad (\epsilon \to 0)$. It follows that

$$\left.\begin{aligned} -\theta'' &= x^{\sigma+n+\epsilon'} && (\sigma+n < -1), \\ -\theta' &= -\mathrm{D} - x^{\sigma+n+1+\epsilon''}, && \\ \theta &= \mathrm{D}x + \mathrm{C} - x^{\sigma+n+2+\epsilon'''} && (\sigma+n+2 < 0) \\ &= \mathrm{D}x + x^{\sigma+n+2+\epsilon'''} && (\sigma+n+2 > 0) \end{aligned}\right\} \quad . \quad (30a)$$

$$\left.\begin{aligned} \theta &= \frac{\mathrm{D}}{\xi} + \mathrm{C} - \xi^{-\sigma-n-2+\epsilon'''} && (\sigma+n+2 < 0) \\ \theta &= \frac{\mathrm{D}}{\xi} + \xi^{-\sigma-n-2+\epsilon'''} && (\sigma+n+2 > 0) \end{aligned}\right\} \quad . \quad (31a)$$

Both cases are consistent with the type form. These are the M-solutions. In either case they depend on two constants of integration, but when $\sigma + n + 2 > 0$ the second must logically be absorbed in the ϵ''' at this stage of the approximation. More exact forms can be obtained by repetition of the argument.

§ 8. *Study of Equation* (B). *Part II. The Existence and Uniqueness of the E-solutions.*—It is already clear that, if the corpus of E-solutions have properties in any way similar to the corpus of E-solutions of

equation (A) when $\sigma + n + 1 = 0$, we shall be able to prove that the M-solutions and the E-solutions are related in the same way as before. This is the astrophysically important theorem. It will be well therefore to take the existence theorem next, as it cannot be handled quite so simply in two stages as for equation (A). Again there is no novelty in the arguments or the result [Lemma II], but I am unaware of a convenient treatment for reference.

Consider therefore the equation

$$y'' + \xi^{-\sigma-n-3}y^n = 0,$$

and attempt to construct solutions $y = y(\xi)$, satisfying $y \sim C\xi$, $y' \sim C$, by Picard's iterative process. To do this we may split the equation into a pair of simultaneous equations by putting $y' = z$ ($z \sim C$), thus

$$\left.\begin{aligned} y' &= z \\ z' &= -\xi^{-\sigma-n-3}y^n \end{aligned}\right\}, \qquad . \quad . \quad . \quad (31b)$$

and solve by iteration with

$$\left.\begin{aligned} y_1 &= C\xi, \qquad z_1 = C, \\ y_s' &= z_{s-1} \qquad\qquad (y_s(0) = 0) \\ z_s' &= -\xi^{-\sigma-n-3}y_{s-1}^n \qquad (z_s(0) = C) \end{aligned}\right\}$$

We can then prove the following series of propositions. First, we can choose ξ_0 independent of s so that if

$$0 \leqslant z_{s-1} \leqslant C, \qquad 0 \leqslant y_{s-1} \leqslant C\xi \qquad (0 \leqslant \xi \leqslant \xi_0),$$

then, for the same ξ_0,

$$0 \leqslant z_s \leqslant C, \qquad 0 \leqslant y_s \leqslant C\xi \qquad (0 \leqslant \xi \leqslant \xi_0) \quad . \quad (32)$$

The proof of this is sufficiently obvious. Secondly, we can prove that the sequences y_s, z_s tend uniformly to limit functions $y(\xi)$, $z(\xi)$ in a range $0 \leqslant \xi \leqslant \xi'_0$. For suppose that

$$|\, z_{s-1} - z_{s-2} \,| < k, \qquad |\, y_{s-1} - y_{s-2} \,| < k\xi.$$

Then, since

$$(y_s - y_{s-1})' = z_{s-1} - z_{s-2},$$
$$|\, y_s - y_{s-1} \,| < k\xi.$$

Also

$$|\, (z_s - z_{s-1})' \,| < \xi^{-\sigma-n-3}n(C\xi)^{n-1} \,|\, y_{s-1} - y_{s-2} \,|,$$
$$< nkC^{n-1}\xi^{-(\sigma+3)},$$
$$|\, z_s - z_{s-1} \,| < \frac{nkC^{n-1}}{-(\sigma+2)}\xi^{-(\sigma+2)},$$
$$< \delta k \qquad (\delta < 1, \xi < \xi_0),$$

where ξ_0 can be fixed when k and C are chosen. Repeating the argument we find that

$$|\, y_{s+1} - y_s \,| < \delta k\xi, \qquad |\, z_{s+1} - z_s \,| < \delta k \quad . \quad . \quad (33)$$

The argument can be repeated indefinitely and the absolute and uniform convergence of the sequences follows. We can prove further, from the first of the relations (33), that

$$| y_s - y(\xi) | < \epsilon\xi \qquad (0 \leqslant \xi \leqslant \xi_0) \qquad . \qquad . \qquad (34)$$

uniformly in this interval, provided that s be great enough.

Thirdly, $y = y(\xi)$, $z = z(\xi)$, so constructed, satisfy the differential equations. For we have necessarily

$$y(\xi) = \operatorname*{Lt}_{s\to\infty} \int_0^\xi z_s d\xi, \qquad z(\xi) = C - \operatorname*{Lt}_{s\to\infty} \int_0^\xi \xi^{-\sigma-n-3} y_s{}^n d\xi.$$

Therefore

$$y(\xi) = \int_0^\xi z(\xi) d\xi + \operatorname*{Lt}_{s\to\infty} \int_0^\xi (z_s - z_s(\xi)) d\xi,$$

$$z(\xi) = C - \int_0^\infty \xi^{-\sigma-n-3} [y(\xi)]^n d\xi + \operatorname*{Lt}_{s\to\infty} \int_0^\xi \xi^{-\sigma-n-3} \{[y(\xi)]^n - y_s{}^n\} d\xi,$$

and with the help of the inequalities (32) and (34) it is easily shown that the limits are zero.

Finally, we can prove that any solution $y = y^*(\xi)$, $z = z^*(\xi)$ of the differential equations (31*b*) with the same initial conditions $y^*(0) = 0$, $z^*(0) = C$, which also satisfies the relation

$$y^*(\xi) = O(\xi),$$

must be identical with the solution we have just constructed. For if Δy, Δz are the differences of the two solutions, $\Delta y(0) = \Delta z(0) = 0$ and

$$(\Delta y)' = \Delta z, \qquad | (\Delta z)' | \leqslant k\xi^{-\sigma-n-3} | \Delta y | y^{n-1},$$
$$\leqslant k'\xi^{-(\sigma+4)} | \Delta y |.$$

In the first place, from the O-relation we know that

$$| (\Delta z)' | \leqslant k''\xi^{-(\sigma+3)}, \qquad | \Delta z | \leqslant \frac{k''}{-(\sigma+2)} \xi^{-(\sigma+2)} < \mu$$

say, provided that $0 \leqslant \xi \leqslant \xi_0$. Therefore by the other equation for the same range

$$| \Delta y | < \mu\xi \qquad (0 \leqslant \xi \leqslant \xi_0).$$

Therefore from the second equation again

$$| \Delta z | < \mu\delta \qquad (\delta < 1),$$

provided only ξ_0 be chosen so that

$$\frac{k'}{-(\sigma+2)} \xi_0{}^{-(\sigma+2)} \leqslant \delta.$$

Without further tampering with ξ_0 we can repeat the process indefinitely, proving in succession that

$$| \Delta y | < \mu\delta\xi, \qquad | \Delta z | < \mu\delta^2, \qquad | \Delta y | < \mu\delta^2\xi, \ . \ . \ .,$$

and therefore that $\Delta y = \Delta z = 0$. Once we are away from the singularity, the unique continuation of this solution is guaranteed by the usual existence and uniqueness theorems. We summarize the results of this section in

Lemma II.—The equation (B) *has a set of proper solutions of the asymptotic form* (30) *or* (31), *and these solutions are the only solutions which satisfy the specified initial conditions and the relationship* $\theta(\xi) = O(1)$.

§ 9. *Study of Equation* (B). *Part III. The General Form of the E-solutions.*—We have already shown, equation (31), that the second approximation to the general E-solution of equation (B) is

$$y = C\xi\left[1 - \frac{C^{n-1}\xi^{-(\sigma+2)}}{(\sigma+1)(\sigma+2)}\right] \quad . \quad . \quad . \quad (35)$$

This is also the result of the second stage of the iterative process. The next stage of the iterative process is equivalent to writing

$$y'' = -\xi^{-(\sigma+3)}C^n\left[1 - \frac{C^{n-1}\xi^{-(\sigma+2)}}{(\sigma+1)(\sigma+2)}\right]^n,$$

integrating twice and adding the proper integration constants which give $C\xi$, the leading term. The form of the result of these operations must be

$$y = C\xi - C^n\xi^{-(\sigma+1)}P_0(C^{n-1}\xi^{-(\sigma+2)}),$$

where $P_0(x)$ is a power series in x, beginning with a non-zero constant term. It follows that

$$y = C\xi P_0(C^{n-1}\xi^{-(\sigma+2)}) \quad . \quad . \quad . \quad (36)$$

so that (35) and (36) are of the same essential form, which has been unaltered by a step of the iteration process. It follows that the final *solution*, constructed as the limit of the iteration process, must have the same form, which we may write

$$y = C\xi f(C\xi^{-\frac{\sigma+2}{n-1}}), \qquad \theta = Cf(C\xi^{-\frac{\sigma+2}{n-1}}) \quad . \quad . \quad (37)$$

The E-solutions therefore are all essentially the same curve, being derivable one from another by mere changes of scale of both co-ordinates, instead of by change of ξ-scale only, as for equation (A).

From the form of the differential equation, we know that the (y, ξ)-solution has $y'' < 0$ so long as $y > 0$, so that its slope is continually diminishing from its value C at $\xi = 0$. It remains only to show that it has a zero. This follows from the form of the solutions of $y'' + \xi^{-\sigma-n-3}y^n = 0$ as $\xi \to \infty$. If $-\sigma - n - 3 = \sigma'$ then $\sigma' + 2 > 0$, $\sigma' + n + 1 > 0$. It therefore follows that as $\xi \to \infty$ the solutions must oscillate,* and therefore there exists a least q such that $f(q) = 0$. From $\xi = 0$ to $C\xi^{-\frac{\sigma+2}{n+1}} = q$, where they first vanish, the Emden solutions all have a steadily decreasing $y'(\xi)$, and consist of a single loop of half-sine-curve type.

* See Fowler, *loc. cit.*, p. 325, Table III.

The E-solutions have an envelope, which should be given by

$$f(C\xi^{-\frac{\sigma+2}{n+1}}) + C\xi^{-\frac{\sigma+2}{n-1}} f'(C\xi^{-\frac{\sigma+2}{n-1}}) = 0.$$

This equation is of the form

$$\frac{d}{d\mu}\{\mu f(\mu)\} = 0 \quad . \quad . \quad . \quad . \quad (38)$$

and must have a unique root. For $\mu f(\mu)$ is the same as $\xi^{\bar{\omega}-1}y$, where we have written $\bar{\omega}$ for $-(\sigma+2)/(n-1)$ for shortness. A root of (38) is equivalent to a root of

$$\frac{d}{d\xi}(z) = \frac{d}{d\xi}(\xi^{\bar{\omega}-1}y) = 0,$$

and z satisfies the differential equation

$$z'' + \frac{2(1-\bar{\omega})}{\xi}z' + \frac{1}{\xi^2}[(-\bar{\omega})(1-\bar{\omega})z + z^n] = 0.$$

Since $\bar{\omega} > 1$, $z'' + 2(1-\bar{\omega})z'/\xi = 0$ only when $z = 0$, and therefore

$$\frac{d}{d\xi}(\xi^{2(1-\bar{\omega})}z') = 0 \quad . \quad . \quad . \quad . \quad (38a)$$

only when $z = 0$. It follows that on the (y, ξ) loop on which z vanishes only at the ends z' vanishes once, and once only, so that there is a unique root of (38); otherwise there would be at least two internal zeros of z' and therefore a root of (38a), where $z \neq 0$. Any E-solution therefore touches this envelope where

$$C\xi^{-\frac{\sigma+2}{n-1}} = \nu,$$

$$\theta = \frac{y}{\xi} = Cf(\nu) = \frac{\nu f(\nu)}{\xi^{-\frac{\sigma+2}{n-1}}} \quad . \quad . \quad . \quad (39)$$

The various E-solutions can cut each other once only (other than at $y = 0$, $\xi = 0$), and no three can go through the same point. This is most easily seen by writing their equations in the form

$$\xi^{\bar{\omega}-1}y = z(C\xi^{\bar{\omega}}).$$

An intersection of two E-solutions means that

$$z(C_1\xi^{\bar{\omega}}) = z(C_2\xi^{\bar{\omega}}), \quad . \quad . \quad . \quad . \quad (40)$$

or that the function z assumes the same value for two different values of its argument. Since the (z, ξ) curve is a simple half-sine curve on which z' vanishes just once, equation (40) can hold for given ξ and C_2 for only one value of C_1 ($\neq C_2$), so that no third E-solution can pass through the intersection of any two. Again the family of (z, ξ) curves are simple half-sine curves identical in form with mere changes of ξ-scale.

They therefore intersect each other once only by the reasoning used for the (y, ξ) curves of equation (A). The order and arrangement of the intersections must also be the same as before. This last point can be made clearer as follows: Consider, for example, the intersections of two members of greater C (C_1, C_2, $C_1 > C_2$) with a given member C_0, and suppose that these intersections are in reversed order. Since for sufficiently great C the intersection with C_0 lies arbitrarily near $\xi = 0$ and therefore then in its proper sequence with the intersection of C_2 on C_0, there must exist a C_3 greater than C_1 or C_2 which passes through the intersection of C_0 and C_2, which is forbidden.

The (z, ξ)-curves being arranged therefore exactly as the (y, ξ) curves of equation (A), multiplication of z by any power of ξ to form the (y, ξ)- or (θ, ξ)-curves of equation (B) distorts the picture, but cannot alter intersections or contacts, and the family of E-solutions have exactly the same properties as for equation (A).

§ 10. *Study of Equation* (B). *Part IV. The Arrangement of the M-solutions.*—The arguments of § 3 can now be repeated almost word for word. We can conclude, just as before, that if a solution starts off from a point P for positive θ between the two E-solutions through that point, then it tends to positive infinity as $\xi \to 0$. It must therefore be one of the M-solutions of asymptotic form (31*a*). We can combine all these results into the following theorems:—

Theorem VI.—As $x \to \infty$ *no proper solution of the equation* (B)

$$\theta'' + x^\sigma \theta^n = 0 \qquad (\sigma + n + 1 < 0),$$

[*or, as* $\xi \to 0$, *of the equation*

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \xi^{-4-\sigma}\theta^n = 0 \qquad (\sigma + n + 1 < 0)],$$

where $n = p/q > 1$, p, q *odd, can oscillate about zero. Such solutions must take one or other of the asymptotic forms*

$$\theta = C - \frac{\alpha C^n x^{\sigma+2}}{(\sigma+1)(\sigma+2)} \qquad (\alpha \sim 1) \quad . \quad [\text{E-}\textit{solutions},$$

$$\theta \sim Dx \quad . \quad . \quad . \quad . \quad . \quad . \quad [\text{M-}\textit{solutions},$$

of which the former are exceptional.

Theorem VII.—Through any point P, ξ_1, *below the envelope* $\theta = m/\xi^{\frac{\sigma+2}{n-1}}$ *of the E-solutions of equation* (B) *pass two E-solutions with slopes* $\alpha, \beta, \alpha > \beta$. (1) *If* $\alpha > \theta'(\xi_1) > \beta$, $\theta = \theta(\xi)$ *is an M-solution and* $\theta(\xi) \to +\infty$ *as* $\xi \to 0$. (2) *If* $\theta'(\xi_1) = \alpha$ *or* $\theta'(\xi_1) = \beta$ *the solution is an E-solution.* (3) *If* $\theta'(\xi_1) > \alpha$ *or* $\theta'(\xi_1) < \beta$ *the solution cuts* $\theta = 0$ *again at a point* ξ_2 *such that* $0 < \xi_2 < \xi_1$.

Any solution through a point P, ξ_1, *above or on the envelope cuts* $\theta = 0$ *again at* ξ_2, *where* $0 < \xi_2 < \xi_1$, *except that through a point* P *on the envelope there passes the single E-solution which touches the envelope there.*

We can now speedily finish off the discussion of equation (B) when p or q is even. When q is even there is nothing to add to the remarks

made on equation (A); at a zero of θ the solution must cease to be a proper solution. When p is even this does not happen, but a crossing of the axis $\theta = 0$ from the positive side as x increases (ξ decreases) must lead to $\theta \to -\infty$. For on putting $\theta = -\phi$ we have the equation

$$\phi'' - x^\sigma \phi^n = 0 \quad . \quad . \quad . \quad . \quad (41)$$

and have to discuss a solution which starts at x_0 with $\phi(x_0) = 0$, $\phi'(x_0) = \delta > 0$. But equation (41) implies that so long as ϕ does not vanish, $\phi'' > 0$, $\phi' \geqslant \delta(x > x_0)$. Therefore $\phi \geqslant \delta(x - x_0)$ and cannot vanish and in fact $\phi \to +\infty$ as stated. So long as $\theta > 0$ all the arguments of the previous case remain unaltered. We summarize in

Theorem VIII.—The equation (B)

$$\theta'' + x^\sigma \theta^n = 0 \quad (\sigma + n + 1 < 0),$$

where $n = p/q > 1$ *and one of* p, q *is even, cannot have a proper solution oscillating about* $\theta = 0$ *as* $x \to \infty$. *For positive values of* θ *proper solutions exist as* $x \to \infty$. *They are monotonic, are arranged in the same way, and have the same asymptotic forms as the corresponding solutions described in Theorems VI., VII.*

§ 11. *Concluding Remarks.*—It obviously remains to discuss the equation $\theta'' + x^\sigma \theta^n = 0$ and its equivalent forms when $\sigma + n + 1 > 0$, though these have not yet proved of astrophysical interest. There are now three cases to consider: $\sigma + 2 > 0$, $\sigma + 2 = 0$, $\sigma + 2 < 0$. The last (if any) is probably the case of astrophysical importance, since usually $\sigma = -4$. When $\sigma + 2 > 0$ and n is odd, all the solutions oscillate; * when n is even all the solutions are of the form $\theta = -x^{-\frac{\sigma+2}{n-1}+\epsilon}$. Fractional indices have not yet been considered, but clearly there can never in such a case be any astrophysically possible solutions with permanently positive θ as $x \to \infty$. When $\sigma + 2 = 0$ the equation can be reduced by the substitution $x = e^t$ to $\theta'' - \theta' + \theta^n = 0$, which has only oscillating solutions when n is odd,† and, when n is even, only monotonic solutions of the type

$$\theta = -(\log x)^{-\frac{1}{n-1}+\epsilon};$$

again nothing is astrophysically possible. The discussion of these cases is already fairly complete. When $\sigma + 2 < 0$ the discussion is still incomplete and one may attempt to push it further by means of the methods of this paper. But it is clear that considerable additional complication is possible, and since this paper has already run to inordinate length, any such attempts must be reserved for a separate communication.

In conclusion, I should record my thanks to Professor Milne, for it was only on hearing from him of the success of his renewed attack on Emden's equation that I recalled the possible relevance of my own previous studies of these equations and was led to attack them afresh, with the results given in this paper.

* Fowler, *loc. cit.*, p. 325, Tables II. and III.

† Fowler, *loc. cit.*, p. 343, Table V.

On a Point-Source Model of a Star. By T. G. Cowling, B.A.

(*Communicated by Professor E. A. Milne.*)

1. At present very little is known about stellar models. In spite of controversies on the subject of stellar structure, very few attempts have been made to investigate and classify the various stellar types corresponding to a given model of a star. The model proposed by Eddington, for which $\kappa\eta$ = const., is the only model which has been investigated with any attempt at completeness: and in the discussion of this model there have been important gaps which are only now being filled. An attempt is made here to give a complete discussion of a particular model.

The model discussed below is a point-source model, *i.e.* a model all of whose energy is supposed to be generated at its centre. It is assumed that κ, the coefficient of opacity, and μ, the mean molecular weight of the gas, are constant through the model. It is not suggested that the model represents physical conditions accurately: but a full discussion of this model should give insight into the nature of stellar structure. The point-source model has been of some importance in considerations of stellar structure.* As Eddington pointed out, the model is a limiting case of stellar structure: it represents the maximum possible concentration of sources of energy towards the centre of the star. In consequence, any result derived from this model should supply an outer limit for results derived from other models. Again, there is reason to believe that the outer layers of a star do not produce any appreciable part of its energy. In consequence, throughout these layers the same laws will govern the variation of density and temperature as prevail in the corresponding layers of a point-source model. A knowledge of the behaviour of density and temperature in the outer layers of a point-source model will therefore be equivalent to a knowledge of their behaviour in the outer layers of a star.

2. It will be assumed that the model is composed of perfect gas. It will also be assumed that at any point there is a hydrostatic radiation-pressure, and local thermodynamic equilibrium. Then we have the equations

$$p = \frac{\mathrm{R}}{\mu}\rho\mathrm{T} \quad . \quad . \quad . \quad . \quad (1)$$

$$p' = \tfrac{1}{3}a\mathrm{T}^4 \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{dp'}{dr} = -\frac{\kappa\mathrm{L}\rho}{4\pi cr^2} \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{dp}{dr} + \frac{dp'}{dr} = -\frac{\mathrm{GM}(r)\rho}{r^2} \quad . \quad . \quad . \quad (4)$$

$$\frac{d\mathrm{M}(r)}{dr} = 4\pi\rho r^2 \quad . \quad . \quad . \quad . \quad (5)$$

* Eddington, *M.N.*, **85**, 408, 1925; *Observatory*, **53**, 208, 1930. Larmor, *Observatory*, **53**, 167, 249, 1930.

where ρ is the density, p, p' are the gas and radiation pressures, T is the temperature, and $M(r)$ denotes the mass enclosed in a sphere of radius r about the centre of the star. L is the luminosity of the source, c the velocity of light, a the radiation constant, R the gas constant, and G the constant of gravitation.

These equations possess the same homology as is met when considering Eddington's model. If r is replaced by r/λ, T by λT, ρ by $\lambda^3\rho$, p by $\lambda^4 p$, p' by $\lambda^4 p'$, then the equations are unaltered. Thus, when one model has been found, an infinite number can be so derived from it. Let us suppose that the stellar matter which was inside a sphere of radius r is made to occupy a sphere of radius r/λ, and let the temperature of the matter in its new position be λ times the temperature of the matter in its former position. Then the new distribution of matter is also a possible one.

3. We now derive a fundamental differential equation whose solution is equivalent to the solution of these five equations. From equations (3) and (4),

$$1 + \frac{dp}{dp'} = \frac{4\pi cG}{\kappa L} M(r) \quad . \quad . \quad . \quad . \quad (6)$$

Thus, differentiating and using equation (5),

$$\frac{d}{dr}\left(\frac{dp}{dp'}\right) = \frac{16\pi^2 cG}{\kappa L}\rho r^2.$$

Now, using equations (1), (2), and (3),

$$-\frac{\kappa L\rho}{4\pi cr^2}\frac{d^2p}{dp'^2} = \frac{16\pi^2 cG}{\kappa L}\rho r^2,$$

$$\therefore \frac{\kappa L}{4\pi cr^2}\frac{d^2}{dp'^2}\left(\frac{R}{\mu}\cdot T\cdot\frac{4\pi cr^2}{\kappa L}\frac{dp'}{dr}\right) = \frac{16\pi^2 cG}{\kappa L}\rho r^2,$$

$$\therefore \frac{1}{T^3}\frac{d}{dT}\left\{\frac{1}{T^3}\frac{d}{dT}\left(r^2T^4\frac{dT}{dr}\right)\right\} = \frac{64\pi^2 acG\mu}{3\kappa LR}r^4. \quad . \quad (7)$$

This equation between T and r is the required differential equation. When it has been solved, (2) gives p', (3) gives ρ, (1) gives p, and (6) gives $M(r)$. Thus it is possible without further integration to obtain a solution of equations (1) to (5) from a solution of equation (7). To each such solution corresponds a special point-source model, though not necessarily a physically possible model.

As the differential equation (7) is of the third order, the general solution involves three constants. The first of these is the constant of homology, choice of which distinguishes a given model from the homologous models. The second was introduced because it was not assumed that $M(0) = 0$. In consequence there may be a point-mass at the centre of the model whose variation will correspond to a variation of the second constant. Lastly, it was found by Eddington when performing his quadratures of a point-source model that, as the integration proceeded outwards, for some models gas-pressure became

small while radiation-pressure was still large, while for other models the converse result occurred. Variation of our third constant will therefore correspond to a variation of the boundary conditions.

4. We proceed to find the solution of equation (7) thus. Let us set

$$r = 1/\theta u \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$\theta^3 = \frac{64\pi^2 acG\mu}{3\kappa LR} \quad . \quad . \quad . \quad . \quad (9)$$

Then the equation takes the form

$$\frac{1}{T^3}\frac{d}{dT}\left\{\frac{1}{T^3}\frac{d}{dT}\left(T^4\frac{dT}{du}\right)\right\} = -\frac{1}{u^4} \quad . \quad . \quad (10)$$

a form which involves none of the physical constants. This equation is homogeneous in T and u, and may therefore be reduced to an equation of the second order. Let

$$T = tu \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Then

$$u\frac{d}{du}\left\{4\left(t + u\frac{dt}{du}\right) + \frac{tu\dfrac{d}{du}\left(t + u\dfrac{dt}{du}\right)}{t + u\dfrac{dt}{du}}\right\} = -t^3\left(u + t\frac{du}{dt}\right).$$

Let also

$$v = u\,.\,\frac{dt}{du} \quad . \quad . \quad . \quad . \quad (12)$$

Then

$$v\frac{d}{dt}\left\{4(t+v) + \frac{tv\left(1 + \dfrac{dv}{dt}\right)}{(t+v)}\right\} = -t^3(t+v) \quad . \quad . \quad (13)$$

The solutions of this second order equation will depend on only two constants of the former three. It is easy to verify that it is the constant of homology that has disappeared in its derivation.

The solutions required are subject to certain restrictions. For, as T and u are positive, t is positive. Also ρ is positive: thus, by equation (3), $\dfrac{dp}{dr}$ is negative: thus $\dfrac{dT}{du}$ is positive—that is, $(t + v)$ is positive. Thus it is only necessary to investigate such solutions, or parts of solutions, as satisfy the conditions that t and $(t + v)$ shall be positive.

5. It does not appear possible to obtain a complete solution of equation (13) in finite terms. Let us therefore seek the forms to which solutions are asymptotic for large or small values of t. If it is first assumed that $v >> t$, $\dfrac{dv}{dt} >> 1$, $\dfrac{d^2v}{dt^2} >> \dfrac{1}{t}$, the equation becomes approximately

$$\frac{d}{dt}\left\{4(t+v) + \frac{tv\left(1 + \dfrac{dv}{dt}\right)}{(t+v)}\right\} = -t^3,$$

whence on integrating

$$4(t+v) + \frac{tv\left(1+\frac{dv}{dt}\right)}{t+v} = -t^4/4 + 4A.$$

Again approximating,

$$4v + t\frac{dv}{dt} = 4A - t^4/4$$

or

$$\frac{d}{dt}(t^4 v) = 4At^3 - t^7/4,$$

whence

$$t^4 v = B + At^4 - t^8/32.$$

Thus, if t is small we may have

$$v \sim B/t^4$$

or

$$v \sim A.$$

If t is large we may have

$$v \sim -t^4/32.$$

As this last asymptotic form is not consistent with the condition that $(t+v)$ is to be positive, it may be disregarded.

Next let it be assumed that $v << t$, $\frac{dv}{dt} << 1$, $\frac{d^2v}{dt^2} << \frac{1}{t}$. Then the differential equation reduces to the ordinary equation

$$4v = -t^4.$$

Thus we have as a possibility for small values of t that

$$v \sim -t^4/4.$$

Lastly, let it be assumed that $v = At + o(t)$, $\frac{dv}{dt} = A + o(1)$, $\frac{d^2v}{dt^2} = o\left(\frac{1}{t}\right)$. It is at once apparent that

$$v = -t$$

is one possible form of this type. If $A \neq 1$, then approximately

$$tA(4+5A) = -t^4(1+A).$$

Thus, if t is small, we may have

$$v \sim -4t/5.$$

Thus, if t is small, v may be asymptotic to any of the forms

$$\frac{B}{t^4}, \quad A, \quad -\frac{4t}{5}, \quad -\frac{t^4}{4}, \quad -t;$$

and if t is large we must have

$$v \sim -t.$$

6. Let us seek more accurate expressions for the solutions which are asymptotic to these forms. It is assumed that there exist series for v in powers of t or of $1/t$, with each of these in turn as leading terms. The series for v are substituted in the differential equation and the expressions are expanded. On equating coefficients, the solutions are found as power series. The results are, for small values of t,

$$\begin{aligned}
&\text{(i)} \quad v = \mathrm{B}/t^4 + \mathrm{A} - 9t/5 - t^4/32 + \ldots \\
&\text{(ii)} \quad v = \mathrm{A} - t - t^4/32 - 13t^5/360\mathrm{A} - 5t^6/144\mathrm{A}^2 + \ldots \\
&\text{(iii)} \quad v = -4t/5 - t^4/112 - 5t^7/12544 + \ldots \\
&\text{(iv)} \quad v = -t^4/4 - t^7/4 + 183t^{10}/256 + \ldots \\
&\text{(v)} \quad v = -t,
\end{aligned}$$

and for large values of t,

$$\text{(vi)} \quad v = -t + 2/t^2 - 3/t^5 + \ldots$$

Of these, (i) depends on two arbitrary constants, and (ii), which is in a sense the limiting form of (i) as $\mathrm{B} \to 0$, depends on one arbitrary constant. It can, however, be shown that for small values of t there is a two-parameter family of solutions asymptotic to (v) which cannot be expressed by a series of powers of t. Let it be assumed that in the neighbourhood of the centre of the model the density is small and the temperature varies very slowly. Then when r is small we may set $\mathrm{T} = \mathrm{T}_0$ in equation (1), and $\mathrm{M}(r) = \mathrm{M}(0)$ in equation (4). We have then

$$p = \frac{\mathrm{R}}{\mu}\rho\mathrm{T}_0,$$

$$\frac{dp}{dr} + \frac{dp'}{dr} = -\frac{\mathrm{GM}(0)\rho}{r^2}.$$

Thus

$$\frac{dp}{dr} = -\left(\mathrm{GM}(0) - \frac{\kappa\mathrm{L}}{4\pi c}\right)\frac{\rho}{r^2}$$

or

$$\frac{d\rho}{dr} = -\frac{\mu}{\mathrm{RT}_0}\left(\mathrm{GM}(0) - \frac{\kappa\mathrm{L}}{4\pi c}\right)\frac{\rho}{r^2}.$$

Set

$$\frac{\mu}{\mathrm{RT}_0}\left(\mathrm{GM}(0) - \frac{\kappa\mathrm{L}}{4\pi c}\right) = -\alpha.$$

Then

$$\frac{d\rho}{dr} = \frac{\alpha\rho}{r^2},$$

whence

$$\rho = \beta e^{-\alpha/r} \qquad . \quad . \quad . \quad . \quad (14)$$

As ρ is to be small near the centre of the star, it is necessary that α shall be positive, *i.e.* that

$$\mathrm{M}(0) < \frac{\kappa\mathrm{L}}{4\pi c\mathrm{G}} \qquad . \quad . \quad . \quad . \quad (15)$$

If this condition is satisfied we shall have approximately

$$\frac{dp'}{dr} = -\frac{\kappa L\beta}{4\pi cr^2}e^{-a/r}.$$

Thus

$$p' = \frac{1}{3}aT_0^4 - \frac{\kappa L\beta}{4\pi ca}e^{-a/r},$$

which is consistent with the condition that the temperature should vary very slowly near the centre. Thus, provided that the condition (15) is satisfied, there exist models such that, near the centre, the density is small and the temperature is finite and varies slowly. This solution corresponds to a solution of the equation (13) such that

$$v \sim -t + \gamma e^{-a\theta T_0/t}.$$

Thus there exists a set of solutions involving two constants, asymptotic to the form (v) for small values of t, which cannot be expressed by means of a series of powers of t. Quadratures of the equation suggest that similarly there are two-parameter families asymptotic to each of the forms (iv) and (vi), and that probably there is a one-parameter family asymptotic to (iii).

7. Next let us investigate the behaviour of solutions in the neighbourhood of the lines $v = 0$ and $v = -t$. Let us consider the behaviour of solutions in the neighbourhood of $v = 0$, $t = t_0$, where t_0 is finite and not zero. The equation (13) can be written in the form

$$\frac{d^2v}{dt^2} = -\frac{t\left(1+\frac{dv}{dt}\right)^2}{v(t+v)} - \frac{(5v+3t)\left(1+\frac{dv}{dt}\right)}{tv} - \frac{t^2(t+v)^2}{v^2}. \quad (16)$$

Thus near this point it becomes approximately

$$\frac{d^2v}{dt^2} = -\left\{\frac{1}{v}\left(1+\frac{dv}{dt}\right)^2 + \frac{2}{v}\left(1+\frac{dv}{dt}\right) + \frac{t_0^4}{v^2}\right\}.$$

If $\frac{dv}{dt}$ is finite when $v = 0$, the equation approximates to

$$\frac{d^2v}{dt^2} = -\frac{t_0^4}{v^2},$$

giving

$$\frac{1}{2}\left(\frac{dv}{dt}\right)^2 = \frac{t_0^4}{v} + \text{const.},$$

which does not give a finite value of $\frac{dv}{dt}$ when $v = 0$. Thus $\frac{dv}{dt}$ cannot be finite when $v = 0$. If $\frac{dv}{dt}$ becomes infinite when $v = 0$, the equation approximates to

$$\frac{d^2v}{dt^2} = -\frac{1}{v}\left(\frac{dv}{dt}\right)^2 - \frac{t_0^4}{v^2}$$

or

$$\frac{1}{2}\frac{d}{dv}\left\{v^2\left(\frac{dv}{dt}\right)^2\right\} = -t_0^4.$$

Thus $$v^2\left(\frac{dv}{dt}\right)^2 = A^2 - 2t_0^4 v.$$

If $A \neq 0$, this approximates to

$$v^2\left(\frac{dv}{dt}\right)^2 = A^2,$$

giving $$v^2 = 2A(t - t_0).$$

If $A = 0$, we have $$(-v)^{1/2}\frac{dv}{dt} = 2^{1/2}t_0^2,$$

giving $$\tfrac{2}{3}(-v)^{3/2} = -2^{1/2}t_0^2(t - t_0)$$

or $$-v^3 = 9t_0^4(t - t_0)^2/2.$$

Thus if the curve corresponding to a solution of equation (13) crosses the line $v = 0$, it crosses at right angles, and $\frac{dv}{dt}$ has opposite signs on opposite sides of the line. As a limiting case, the curve may have a cusp on the line, whose cuspidal tangent is at right angles to the line, and such that v is negative near the cusp.

Consider next the behaviour of the curve near the point $v = -\alpha$, $t = \alpha$. The equation then approximates to

$$\frac{d^2v}{dt^2} = \frac{\left(1 + \frac{dv}{dt}\right)^2}{(t + v)} - \frac{2\left(1 + \frac{dv}{dt}\right)}{t}$$

or, if $\frac{dv}{dt}$ does not approach -1 as $(v + t)$ becomes small, to

$$\frac{d^2v}{dt^2} = \frac{\left(1 + \frac{dv}{dt}\right)^2}{t + v}.$$

Thus $$1 + \frac{dv}{dt} = A(t + v),$$

whence $$\log(t + v) = At + B.$$

This cannot be consistent with $t = -v = \alpha$. We are left with the possibility that $\frac{dv}{dt}$ does tend to -1 as limit. In this case, the above again holds unless

$$\frac{dv}{dt} + 1 = O(t + v).$$

In this case the differential equation gives

$$\frac{d^2v}{dt^2} = O(t + v),$$

and successive differentiations of the equation give

$$\frac{d^3v}{dt^3} = O(t + v), \text{ etc.}$$

Thus $(t + v)$ and all its derivatives vanish at the point. Hence the only solution which passes through the point is $v = -t$. Thus no curves corresponding to solutions of the equation can cross the line $v = -t$.

Equally there are no solutions for which v is infinite while t is finite. This can be seen on using the analysis by which were investigated the solutions for which $v >> t$. In consequence, the behaviour of solutions of the equation at all points of the boundary of the region for which t and $(t + v)$ are positive is now known. Their behaviour at internal points has now to be considered.

8. From (16) it follows that, when $\frac{dv}{dt} = -1$,

$$\frac{d^2v}{dt^2} = -\frac{t^2(t+v)^2}{v^2},$$

and so, unless $t = 0$ or $t + v = 0$, $\frac{d^2v}{dt^2}$ is negative. That is, in the interior of the region considered, $\frac{dv}{dt}$ cannot increase through the value -1 as t increases: it can only decrease through this value.

Now let us consider the types of solution which are asymptotic to the solution (vi) of section (6) for large values of t. As t decreases, either $\frac{dv}{dt}$ becomes greater than -1 or it does not. If it does become greater than -1, it cannot again become less, by what we have just proved. Hence, since $\frac{dv}{dt}$ must become infinite and negative for the curve to cross the line $v = 0$, the curve does not cross this line. Neither can it cross the line $v = -t$. Hence it must ultimately become asymptotic to one of the solutions (iii), (iv), and (v).

Next let us suppose that $\frac{dv}{dt}$ does not become greater than -1. Then the curve crosses the line $v = 0$, or, in the limit, has a cusp on it. If it has a cusp on it, then after the cusp $\frac{dv}{dt}$ becomes positive, and so is greater than -1. Hence, as above, the solution must ultimately become asymptotic to one of the solutions (iii), (iv), and (v). If the curve crosses the line $v = 0$, then, as we proceed along the curve, t is increasing and $\frac{dv}{dt}$ is positive. While $\frac{dv}{dt}$ remains greater than -1, $\frac{d^2v}{dt^2}$ is, by equation (16), less than $-t^2$, and so is less than a negative constant. Thus $\frac{dv}{dt}$ will decrease below the value -1. As it cannot

increase through this value, the curve must recross the line $v = 0$. Now t begins to decrease, and $\frac{dv}{dt}$ is positive and so greater than -1.

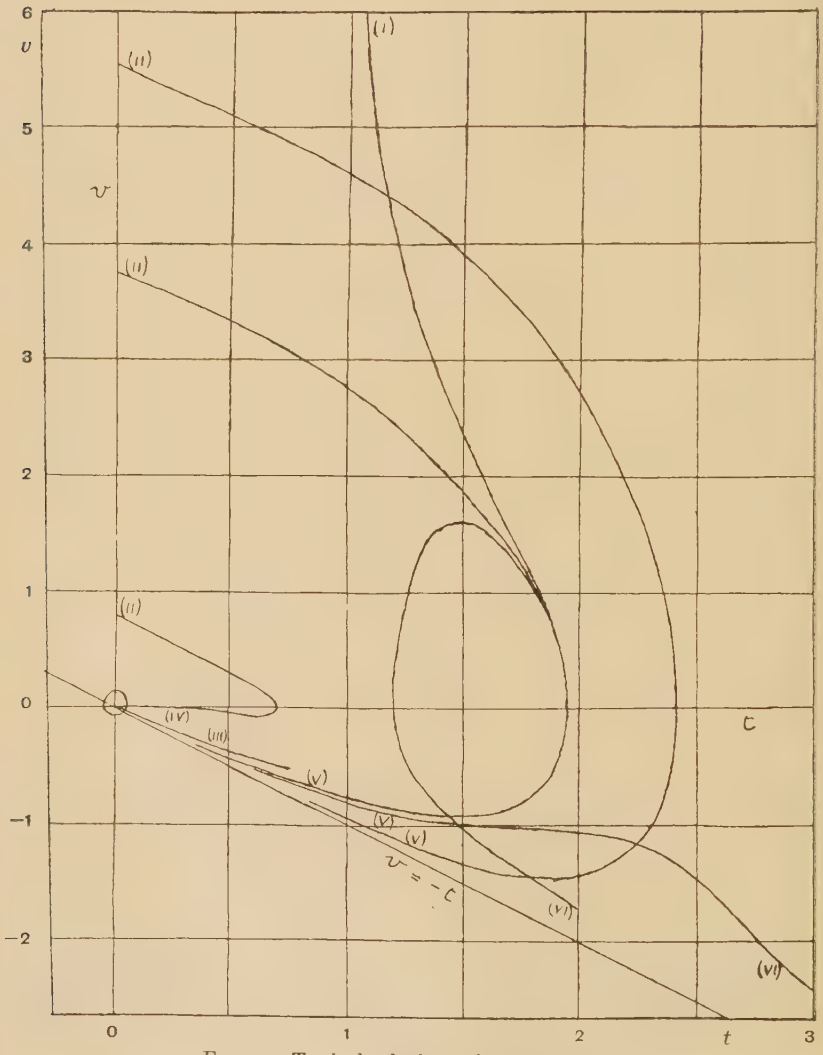

FIG. 1.—Typical solutions of equation (13).

Hence, as above, the solution must become asymptotic to one of (iii), (iv), and (v).

Next let us consider the solutions which begin with one of the forms (i) and (ii). In this case, when t is small, $\frac{dv}{dt} < -1$. Hence, as t

increases, $\frac{dv}{dt}$ will remain less than -1, and so v will decrease to zero. The curve crosses the line $v = 0$; t begins to decrease and $\frac{dv}{dt}$ becomes positive. Hence, by the preceding arguments, the solution will become asymptotic to one of the forms (iii), (iv), and (v).

This exhausts all the possible forms of solution. Curves corresponding to typical solutions of these types are shown in fig. 1.

9. We now investigate the physical properties of the models which correspond to the solutions just obtained. From equation (12) we derive

$$\log\left(\frac{u}{u_0}\right) = \int\frac{dt}{v},$$

where u_0 is a constant. On using equation (8) this takes the form

$$\log\left(\frac{r}{r_0}\right) = -\int\frac{dt}{v}.$$

On substituting the forms (i)–(vi) in this equation, the following results are derived. The solutions (iii), (iv), and (v) apply near the centre of the model. The solutions (i) and (ii) apply just inside a sphere of radius r_0. The solution (vi) applies for large values of r, that is, at large distances from the centre. There is a twofold classification of solutions: they are asymptotic to one of (iii), (iv), and (v), and to one of (i), (ii), and (vi). This twofold classification corresponds to a twofold classification of models, according to their behaviour at the centre and at the boundary.

We first consider the classification according to the behaviour at the centre. Solution (iii) corresponds to a model with a point-mass equal to $\kappa L/4\pi cG$ at the centre. Near the centre the density is small, the temperature is large: gas-pressure is finite and radiation-pressure infinite.

Solution (iv) also corresponds to a model with a central point-mass equal to $\kappa L/4\pi cG$. Near the centre the density is large, and the temperature is large: in consequence gas and radiation pressures are also large.

Solution (v) corresponds to a model with any positive or negative point-mass less than $\kappa L/4\pi cG$ at the centre. Near the centre the density is small, and the temperature is finite: gas-pressure is small and radiation-pressure finite.

Next we consider the classification of models according to their behaviour at the boundary. It is hardly correct to speak of the boundary of a model corresponding to solution (vi), as such a model occupies the whole of space, and has infinite mass. For a model of this type, as the distance from the centre becomes large, temperature tends to a finite limit, and density tends to zero: $M(r)$ becomes infinite: and gas and radiation pressures tend respectively to zero and non-zero limiting values.

Models corresponding to the other solutions are finite. Those corresponding to solution (i) have zero temperature and infinite density at the boundary. The mass of the model can have any value: gas-pressure is finite and radiation-pressure zero at the boundary. As the models demand a finite gas-pressure at the boundary, they can only exist if surrounded by a constraining surface.

The models corresponding to the solution (ii) have zero temperature and density at the boundary: in consequence both radiation and gas pressures vanish at the boundary. These models can have any mass greater than $\kappa L/4\pi cG$.

10. Let us now select those models whose boundary conditions resemble those found in actual stars. At the boundary of a star density becomes small, while radiation-pressure tends to a value which, though small compared with the interior value, is not zero. These conditions are satisfied by some models corresponding to the solution (vi). Here a difficulty arises: for these models are infinite models with infinite masses.

This difficulty has been met before. Milne * has observed that Eddington's model "$\kappa\eta$ = const." fails to terminate if account is taken of the fact that radiation-pressure does not vanish at the boundary: and it is not difficult to show that in this case likewise the mass of the model is infinite. It may be supposed that the difficulty is a consequence of the approximate nature of equation (3), which becomes invalid in a stellar reversing layer. It is therefore pertinent to observe that McCrea,† in his paper on the Spherical Chromosphere, was only able to avoid the same difficulty by an *ad hoc* assumption as to the variation of the temperature through the chromosphere.

We are therefore led to regard actual stars as non-terminating, and to consider them as fading into the background of interstellar matter. But it is known that they do effectively terminate—that is, that at a certain level density becomes negligibly small. This property is also possessed by our models. At large distances from the centre, the density is given approximately by the equation

$$\rho = \frac{RT_0}{2\pi G\mu} \cdot \frac{1}{r^2},$$

where T_0 is the temperature at an infinite distance from the centre. If we adopt the values $T_0 = 10{,}000°$, $r = 10^{11}$ cm., $\mu = 2{\cdot}3$, this gives for the value of the density, $\rho = 8{\cdot}634 \,.\, 10^{-5}$ gr./cm.^{3}, which is negligible in comparison with the density in the interior, which is of the order of 1 gr./cm.^{3}. Thus the model does effectively possess a boundary.

The model may be given a suitably defined photosphere. We define the photosphere as the level whose temperature is equal to the effective temperature. Thus at the photosphere

$$L = \pi acr^2T^4.$$

* *M.N.*, **90**, 22, 1929.
† *M.N.*, **88**, 733, 1928.

If the model is deformed into a homologous model, r^2T^4 is not unaltered. Thus the new photosphere will not be at the level which corresponds to the former photospheric level. The mass enclosed by the photosphere will not be altered appreciably, however, if the range of variation of the constant of homology is restricted to such values as make the photosphere lie in the boundary region : for in the boundary region the density is small.

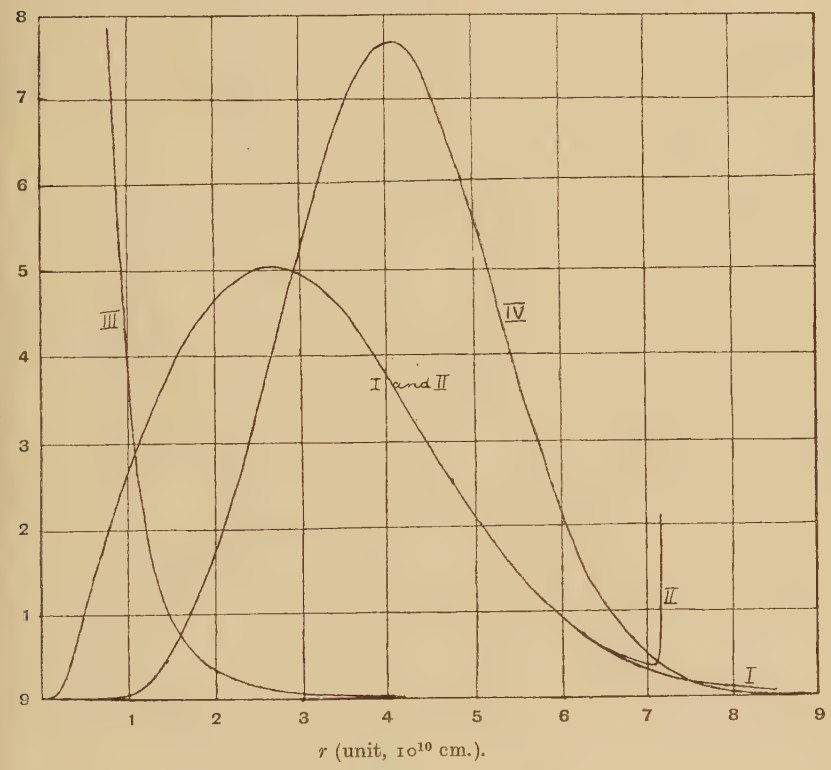

Fig. 2.—Density curves for models I.–IV.

It is of interest to discover what effect variation of the effective temperature has on the conditions in the interior. Eddington * suggests that the effect is negligible, but it is worth while to investigate the matter fully for this model. The question to be answered is the following :—The magnitudes of the central point-source and point-mass and the value of the central temperature being assigned, what alteration in the conditions in the interior is produced by a variation of the photospheric temperature within the limits of stellar effective temperatures ?

* *Internal Constitution of the Stars*, p. 127.

This has been investigated by numerical integration. We adopt the values $\kappa = 10/\text{grs.} \times \text{cm.}^2$, $L = 3{\cdot}78 \cdot 10^{35}$ ergs/sec., $\mu = 2{\cdot}3$, $M(0) = 0$, and take 40,000,000° as the central temperature. The solution proceeds by quadrature from the centre outwards: for small values of r the density varies in accordance with equation (14). For large values of β a terminating model with finite gas-pressure at the

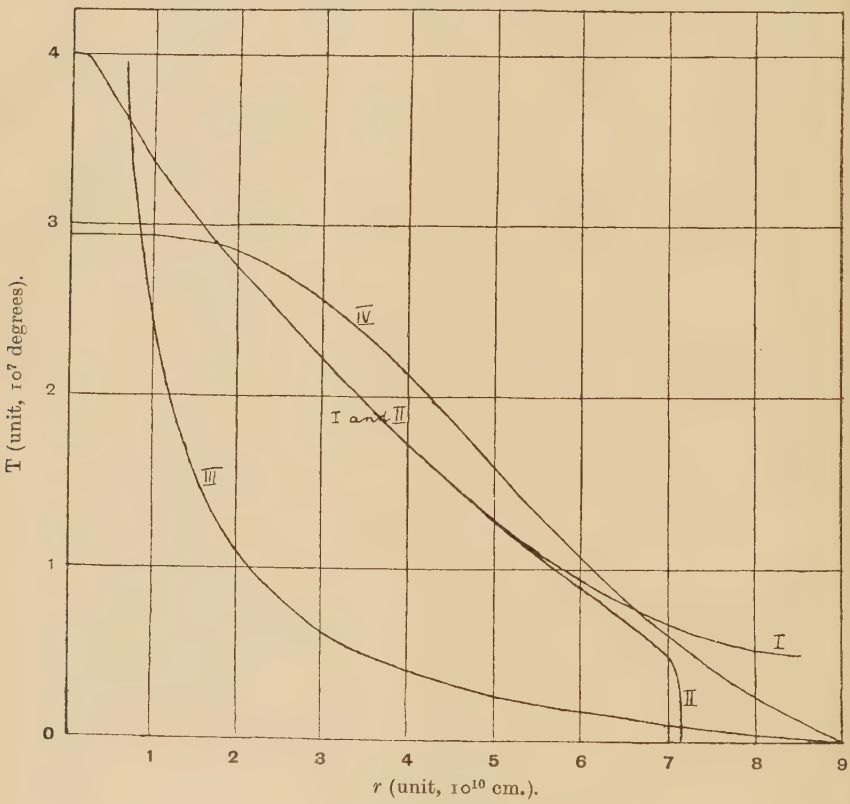

FIG. 3.—Temperature curves for models I.–IV.

boundary is obtained: for small values of β an infinite model is obtained. The value of β is adjusted till a suitable model is arrived at. In our work two models were finally obtained. The first is an infinite model whose temperature at infinity is about 4,000,000°: the second is a finite model. Tables of the values of temperature and density at various distances from the centre are appended. These tables show that the temperatures of the two models are not seriously different save near their boundaries, and that the densities only diverge appreciably very near the boundaries. Since the divergence of these two models is far more than the divergence of models whose effective

temperatures lie within the limits of stellar effective temperatures, it may be concluded that, except at points very near the boundary, the value of the effective temperature has a negligible effect on conditions in the interior. This is Eddington's conclusion. There is, however, a big difference between his point of view and that which we have had to adopt in that he supposed his models to possess finite mass : in our considerations the " mass of the star " has to be replaced by the " mass inside the photosphere."

Table I.

	(1) Infinite Model.			(2) Finite Model.		
r.	T.	ρ.	M(r).	T.	ρ.	M(r).
0	40	0	0	40	0	0
0·1	39·98	0·0042	0·00000	39·98	0·0042	0·00000
0·2	39·71	0·1295	0·00002	39·71	0·1296	0·00002
0·5	37·41	1·136	0·00036	37·41	1·137	0·00036
1·0	33·47	2·722	0·00818	33·47	2·723	0·00818
1·5	30·21	3·874	0·04209	30·20	3·876	0·04210
2·0	27·28	4·643	0·12848	27·27	4·646	0·12851
2·5	24·55	5·018	0·28166	24·53	5·022	0·28180
3·0	21·95	4·945	0·5207	21·92	4·950	0·5211
3·5	19·43	4·509	0·8354	19·39	4·515	0·8361
4·0	17·08	3·749	1·2036	17·02	3·756	1·2046
4·5	14·85	2·897	1·5800	14·75	2·905	1·5813
5·0	12·81	2·087	1·9307	12·63	2·095	1·9330
5·5	10·97	1·414	2·230	10·66	1·422	2·234
6·0	9·35	0·9055	2·466	8·81	0·9149	2·472
6·5	7·96	0·5509	2·641	6·95	0·5619	2·650
6·8	7·25	0·3992	2·720	5·68	0·4163	2·730
7·0	6·83	0·3190	2·763	4·50	0·3545	2·776
7·1	6·63	0·2845	2·781	3·38	0·3693	2·798
7·14	6·56	0·2718	2·788	0	∞	2·809
7·5	5·96	0·1762	2·842			
8·0	5·37	0·0936	2·891			
8·5	5·02	0·0489	2·920			

The units used in the tables are as follows : The unit of r is 10^{10} cm. ; that of T, 10^6 degrees ; that of ρ, grms./cm.^{3} ; and that of M(r), 10^{33} grs.

11. From what has been proved we conclude that, except in regions very near the boundary, conditions in the models whose boundary temperatures lie within the limits of stellar effective temperatures differ very slightly from conditions in models which have zero boundary temperature and density—that is, models corresponding to solution (ii) of equation (13). Our further investigations will accordingly be limited to a consideration of models of this type. Such models are homologous to a single infinity of models.

Near the boundary of such a model we shall have

$$\begin{aligned}\frac{dT}{du} &= t + v \\ &= A - t^4/32 \ . \ . \ . \\ &\doteq A.\end{aligned}$$

Thus, approximately,

$$\begin{aligned}T &= A(u - u_0) \\ &= \frac{A}{\theta}\left(\frac{1}{r} - \frac{1}{r_0}\right).\end{aligned}$$

Hence, from (2), (3), and (1),

$$\begin{aligned}p' &= \frac{a}{3}\frac{A^4}{\theta^4}\left(\frac{1}{r} - \frac{1}{r_0}\right)^4 \\ \rho &= \frac{16\pi ca}{3\kappa L}\frac{A^4}{\theta^4}\left(\frac{1}{r} - \frac{1}{r_0}\right)^3 \qquad . \ . \ . \qquad (17) \\ p &= \frac{16\pi caR}{3\kappa L\mu}\frac{A^5}{\theta^5}\left(\frac{1}{r} - \frac{1}{r_0}\right)^4.\end{aligned}$$

Hence, if M is the mass of the model,

$$\begin{aligned}GM - \frac{\kappa L}{4\pi c} &= \frac{\kappa L}{4\pi c}\left(\frac{dp}{dp'}\right)_{r=r_0} \\ &= \frac{4R}{\mu}\frac{A}{\theta} \\ &= A\left(\frac{3R^4\kappa L}{\pi^2 acG\mu^4}\right)^{1/3} \qquad . \ . \ . \qquad (18)\end{aligned}$$

Thus variation of A from 0 to ∞ will correspond, when the luminosity is fixed, to a variation of mass from $\kappa L/4\pi cG$ to ∞, and, when the mass is fixed at the value M, to a variation of luminosity from $4\pi cGM/\kappa$ to zero. We wish now to discover for which values of A conditions at the centre correspond to each of the solutions (iii), (iv), and (v).

Reference to fig. 1 suggests that the curves corresponding to solutions of the type (ii) will probably surround one another and not cross each other. If this is so, then, for an assigned value of L, a light model will have at its centre a point-mass equal to $\kappa L/4\pi cG$ surrounded by a dense region, and a heavy model will have at its centre a smaller point-mass surrounded by a region in which the density is small. It does not appear possible to give a rigorous proof of this: but the following considerations appear to leave little doubt that this is the actual case.

Near the boundary we have, from (17) and (18),

$$\rho = \frac{\pi ac\mu^4}{48\kappa LR^4}\left(GM - \frac{\kappa L}{4\pi c}\right)^4\left(\frac{1}{r} - \frac{1}{r_0}\right)^3.$$

Hence, if we consider two models of the same radius and luminosity, but different masses, the percentage of the mass lying in a given region

near the boundary will be smaller for the lighter model than for the heavier. Hence it appears probable that the greater part of the mass of a light model must be concentrated at its centre, while so much of the mass of a heavy model lies in its outer layers that a negative central point-mass is needed to balance it.

So that there may be no doubt as to the validity of these conclusions, the matter has been investigated by numerical integration. The work of the previous section shows that, with the values of κ, L, μ adopted there, the model with zero central point-mass has a mass of about $2{\cdot}9 \cdot 10^{32}$ grs. Two other models have been considered, whose masses are respectively $0{\cdot}75275 \times 10^{32}$ grs. and $4{\cdot}159 \times 10^{32}$ grs., *i.e.* are respectively less than and greater than the above. The first of these has a dense region at the centre, surrounding a point-mass $\kappa L/4\pi cG$: the second has a central point-mass equal to $-8{\cdot}1486 \times 10^{31}$ grs., surrounded by a region through which the density is small (see Table II.). It may therefore be concluded that a model of large mass has less mass at and near the centre than a model of small mass.

The matter may be put thus. The heavy models are only prevented from collapsing by a strong repulsive force from the centre due to radiation-pressure, supplemented perhaps by a gravitational repulsion due to the presence of a negative central point-mass: the light models are only held together against radiation-pressure by the gravitational attraction of a dense centre.

Table II.

	(3) Light Model.			(4) Heavy Model.		
r.	T.	ρ.	M(r).	T.	ρ.	M(r).
0	∞	∞	0·15055	29·35	0	−0·8486
0·25	96·36	211·1	0·6428	29·35	0·0000	−0·8486
0·5	48·94	29·36	0·6732	29·35	0·0001	−0·8486
1·0	23·84	3·586	0·7052	29·34	0·0753	−0·8484
1·5	15·11	0·9374	0·7226	28·96	0·5903	−0·8464
2·0	10·65	0·3335	0·7322	28·19	1·7154	−0·8236
2·5	7·947	0·1393	0·73995	27·04	3·443	−0·7421
3·0	6·128	0·0642	0·74441	25·45	5·360	−0·5340
3·5	4·825	0·0313	0·74750	23·50	6·969	−0·1247
4·0	3·839	0·0159	0·74947	21·15	7·660	+0·5441
4·5	3·075	0·0082	0·75078	18·54	7·069	1·159
5·0	2·461	0·0042	0·75162	15·80	5·518	2·267
5·5	1·960	0·0021	0·75215	13·10	3·676	2·852
6·0	1·542	0·0010	0·75246	10·54	2·113	3·443
6·5	1·186	0·0005	0·75263	8·224	1·060	3·820
7·0	0·881	0·0002	0·75271	6·151	0·4561	4·041
7·5	0·617	0·0001	0·75274	4·314	0·1600	4·108
8·0	0·386	0·0000	0·75275	2·705	0·0394	4·151
8·5	0·181	0·0000	0·75275	1·274	0·0041	4·158
9·0	0	0	0·75275	0	0	4·159

12. It is now necessary to review our results from the two points of view suggested at the beginning of this paper. First, our investigation can be regarded as an investigation of conditions in the outer layers of a star. When this standpoint is adopted, no models can be rejected because the conditions at the centre are not physically possible. We can only assert that, for models at whose centres physically impossible conditions do prevail, the above analysis must cease to be accurate in some region surrounding the centre. Modifications in the gas laws may occur, and active matter must appear. The investigation of the nature of these modifications is outside the scope of this paper. Outside the region in which these modifications occur, the variation of temperature and density will follow the lines illustrated in figs. 2 and 3.

Secondly, we regard our investigation as being one of a limiting case of stellar equilibrium. In this case an impossible condition of affairs was assumed when it was assumed that all the sources of energy were concentrated at the centre. Any absurdity in the density distribution near the centre cannot be regarded as a reason for the rejection of the model. If there is a positive point-mass at the centre of the model, its magnitude is not greater than $\kappa L/4\pi cG$, and so, as the concentration of sources is decreased, this will decrease. Thus a positive central point-mass may be regarded as a consequence of the point-source : no models with a positive central point-mass may be rejected. Thus any mass in a certain range is consistent with the given luminosity.

From a study of this model no evidence has been found for a belief that to a given luminosity corresponds a unique mass. Our investigations suggest that, in the absence of a limiting assumption as to the mode of generation of energy, all masses in a certain range may be associated with a given luminosity.

Summary.

1. A special point-source model of a star, for which the coefficient of opacity is assumed to be constant, is discussed, and the various types of model are classified.

2. It is shown that the models which satisfy boundary conditions consistent with facts derived from actual stars are infinite in size and mass, but that they effectively terminate.

3. It is shown that it is impossible to deduce any evidence about a mass-luminosity law from such a model.

In conclusion, I should like to express my thanks to Professor Milne, to whose advice this piece of work has owed a great deal since it was begun at his suggestion last December.

The Opacity of Extended Stellar Envelopes.

By Sir Arthur Eddington, F.R.S.

1. *Extended Envelopes.*—Algol and Capella have nearly the same mass (about 4·2 ⊙), and in accordance with the mass-luminosity relation they have nearly the same luminosity, but the radius of Algol is 2,000,000 km. and the radius of Capella 9,500,000 km. It is inferred that Algol is in a more condensed state than Capella, and we accordingly believe that it is in a later stage of evolution. We ought not, however, to overlook the possibility that the difference of radius may be a very superficial matter. Consider an outer shell or envelope of a star containing only a small fraction of the mass; it would be possible, whilst leaving the condensation of the main part of the mass unchanged, to puff out the envelope of Algol so as to extend to the radius of Capella, or the even greater radius of a K-M giant. Since only a small part of the mass would be affected, the formation or collapse of an extended envelope of this kind would involve relatively little change of potential energy, and no special evolutionary significance could be attributed to it. On this view, instead of recognising a giant stage preceding the main series stage in evolution, we should regard giantism as a casual phenomenon, perhaps like Cepheid variability.

To fix ideas I shall define the *envelope* of a star to be the region containing the outermost $\frac{1}{10}$ of the mass, the inner $\frac{9}{10}$ constituting the core or "real star." The suggestion is that the differences of observed radius may be largely differences of extension of the envelope, the "real star" being the same. It is to be remembered that the envelope is in any case likely to have considerable extension; thus in the most familiar model, represented by the polytrope $n = 3$, the envelope (as defined above) occupies the outer half of the radius and seven-eighths of the whole volume. Moreover, it has been recognised that the actual extension of the envelope may differ considerably from the model, and an extension over, say, $\frac{2}{3}$ instead of $\frac{1}{2}$ the radius would scarcely be counted abnormal.* I reserve the term "extended envelope" for a more exaggerated inflation—when the observed radius is 5, 10, or a greater multiple of the radius of the core. This gross inflation was first seriously urged by Sir J. H. Jeans; † he regarded it as a feature of the constitution of stars of large mass.

Extended envelopes play a prominent part in Jeans's liquid star theory and in Milne's new nuclear star theory. Milne's theory in all cases and Jeans's theory in some cases demands a large concentration of mass at the centre of the star, with a consequent reduction of density in the outer regions. It is not easy to discover the numerical value of the extension contemplated by them, but inasmuch as they postulate central densities some thousands of times greater than those given by the polytrope $n = 3$ we are presumably not exaggerating if we take the outermost $\frac{1}{10}$ of the mass to extend over $\frac{4}{5}$R in their models instead

* *Internal Constitution of the Stars*, p. 96.

† *Monthly Notices*, **87**, 40, 1926.

of the normal $\frac{1}{2}$R.* My chief numerical results refer to an extension of this amount. Results for smaller or greater extensions could easily be calculated.

The present paper is largely devoted to the theories of Jeans and Milne, because it is in regard to them that I have reached positive results. I show that so far from curing the discrepancy between astronomical and physical calculations of the opacity of stellar material (which is the outstanding difficulty in my own theory), the other theories give a greater discordance. The disagreement of the opacity remains a mystery. Hopes that wave-mechanics would lead to a reconciliation seem to have faded away. The obstinacy of the high astronomical value is now further exemplified by the fact that it persists in theories so diverse as those of Jeans, Milne, and myself. Apart from the one (not very acceptable) loophole that I have pointed out, viz. that stellar material contains a large admixture of hydrogen, there seems to be no known way of meeting the difficulty.

My primary interest in the investigation was, however, in the possibility that the recognition of extended envelopes might be incorporated as a development in my own theory. It is a rather attractive speculation that giant stars differ from main series stars of the same mass only in the extension of the envelope, the core within the envelope being the same. Up to a certain mass (about 2·5) all stars except white dwarfs form a single sequence in the Hertzsprung-Russell diagram; but above mass 2·5 the properties are no longer determined simply by the mass, and we have giants of various spectral types as well as the main series. The suggestion is that, disregarding a merely superficial difference of extension of the envelope and considering only the "real stars," they continue to form a single sequence; and, for example, the result that in main series stars the central temperature remains practically the same throughout the sequence would become extended to all stars other than white dwarfs. This suggestion would be supported if we could show that the extension of the envelope becomes very sensitive to small changes of the conditions (chemical composition, etc.) as the mass increases above 2·5.

To apply this idea to Algol and Capella we must suppose that Algol (being a main series star) has a fairly normal envelope. It consists therefore of a core of radius 1,000,000 km., with an envelope extended to 2,000,000 km. Capella is to have a similar core, with an envelope extended to 9,500,000 km.; the ratio of the observed radius to the core radius is thus $9\frac{1}{2}$ to 1. We may say briefly that the *extension* of Algol is 2 and that of Capella $9\frac{1}{2}$. Our hypothesis will require that the redder giants have still greater extension.†

* Jeans gives the central density of Betelgeuse as 8000 times that of the polytropic model (*Nature*, 1928 Feb. 25, p. 279). I do not know what would be his view as to the corresponding ratio for Capella, which is the example considered in this paper, since it depends on his arbitrary assumptions as to atomic volume.

† It will be seen that this hypothesis differs from that of Jeans, although there is some qualitative resemblance. In Jeans's theory giantism is partly due to inflation of the envelope; but to admit (as we here suggest) that it is *wholly* due

I think we are not yet in a position to decide for or against the hypothesis here suggested, but the result of our investigation (§ 8) is by no means favourable to it.

2. *General Formulæ for the Envelope.**—According to the definition of the envelope, the mass M_r interior to a point r cannot differ more than 10 per cent. from M. Hence the potential at a point in the envelope is (to within 10 per cent.)

$$\phi = GM\left(\frac{1}{r} - \frac{1}{R}\right) \quad . \quad . \quad . \quad . \quad (1)$$

where as usual ϕ is measured from the surface of the star. If the heat supply of the star is wholly interior to the envelope the outflow of radiation L_r across a sphere of radius r is constant and equal to L; then $L_r/M_r = L/M$ to within 10 per cent. If the envelope contributes to the heat supply, L_r/M_r is still more nearly constant. Hence (to within 10 per cent.)

$$H/g = L_r/4\pi GM_r = L/4\pi GM \quad . \quad . \quad . \quad (2)$$

The formulæ (1) and (2) lead to a great simplification in the theory of the envelope; it is to be remembered that they are not applicable to the rest of the star.

We take the law of opacity to be

$$k = \kappa\rho/\mu T^v = \kappa'\rho/\mu T^3 \quad . \quad . \quad . \quad (3)$$

where μ is the molecular weight, κ is a constant (the *intrinsic* opacity), and v, according to physical theory, is expected to be about $3\frac{1}{2}$. By (3) $\kappa' = \kappa T^{3-v}$, so that κ' is a slowly varying function of T; it is useful in connection with the following formulæ, which in various equivalent forms have been given by a number of writers:—

Write

$$\sigma = \frac{a}{3\mathfrak{R}}\frac{L\kappa'}{4\pi cGM} \quad . \quad . \quad . \quad . \quad (4)$$

Then there is an "equilibrium" value of the ratio of radiation-pressure to whole pressure $1 - \beta$ given by

$$(1 - \beta)^2/\beta = \sigma \quad . \quad . \quad . \quad . \quad (5)$$

and Woltjer and others have shown that the effect of any end-disturbance from this equilibrium value rapidly dies out. In any extended region of constant or slowly varying σ, $1 - \beta$ will approximately attain its equilibrium value at every point. If κ is prescribed, a rough idea of the temperature of the region is sufficient to give a reasonably good estimate of κ', and we can then at once determine $1 - \beta$ from (4) and (5). This is very useful when we want to form a preliminary idea of the condition in the envelope as regards radiation-pressure.

to this inflation would be contradictory to his theory of evolution of the core through successive zones of stability, jumping to higher and higher density as successive rings of electrons are ionised.

* The notation (except where otherwise defined) is that of *Internal Constitution of the Stars*, chap. vi.

It seems obvious that the formation of extended envelopes will occur when the material is distended by radiation-pressure greater than that allowed for in the polytropic model. By (4) and (5) this requires that κ' should be greater. For this reason we should expect that theories which postulate an extended envelope cannot afford to adopt a lower intrinsic opacity than those which do not. This is borne out by the detailed calculations which follow.

A point of importance brought out in the succeeding calculations is that the general temperature within the extended envelopes is comparable with and in some cases (§ 4) even higher than the temperature assigned to the main part of the star on the ordinary polytropic model. Had the temperature turned out low, the high opacity might have been excused on the ground that the physical theory of absorption (which gives much too low an opacity) has been worked out and tested primarily for high temperatures, so that a failure below, say, 500,000° is not very serious. Actually, however, the discordance given by my theory of the main interior and the larger discordance given by the hypothesis of extended envelopes refer to much the same temperature.

3. *Stars with Low Radiation-Pressure.*—The simplest case is when radiation-pressure is negligibly small compared with gas-pressure. We therefore consider first a star like Krueger 60, which on Milne's theory has an extended envelope. Since we are treating the outer part of the star it is desirable to allow for a variation of molecular weight; we therefore set

$$\mu = \mu_0 \mathrm{T}^l \quad . \quad . \quad . \quad . \quad (6)$$

Assume for trial

$$\rho = \rho_0 \mathrm{T}^s \quad . \quad . \quad . \quad . \quad (7)$$

where ρ_0 and s will presently be determined. By hypothesis the total pressure P is approximately the same as the gas-pressure, so that

$$\mathrm{P} = (\mathfrak{R}/\mu)\rho\mathrm{T} = (\mathfrak{R}\rho_0/\mu_0)\mathrm{T}^{s+1-l}.$$

Hence

$$d\mathrm{P} = (s + 1 - l)(\mathfrak{R}\rho_0/\mu_0)\mathrm{T}^{s-l}d\mathrm{T}$$

and

$$d\phi = d\mathrm{P}/\rho = (s + 1 - l)(\mathfrak{R}/\mu_0)\mathrm{T}^{-l}d\mathrm{T} \quad . \quad . \quad (8)$$

Hence, integrating,

$$\mathrm{T}^{1-l} = \frac{1-l}{s+1-l}\frac{\mu_0}{\mathfrak{R}}\phi,$$

and by (6)

$$\mathrm{T} = \frac{1-l}{s+1-l}\frac{\mu\phi}{\mathfrak{R}} \quad . \quad . \quad . \quad . \quad (9)$$

Again, since

$$\mathrm{H} = -\frac{ac}{3k\rho}\frac{d\mathrm{T}^4}{dr}, \qquad g = -\frac{d\phi}{dr},$$

we have

$$\frac{\mathrm{H}}{cg} = \frac{4}{3}\frac{a\mathrm{T}^3}{k\rho}\frac{d\mathrm{T}}{d\phi} = \frac{4}{3(s+1-l)}\frac{a{\mu_0}^2}{\kappa{\rho_0}^2\mathfrak{R}}\mathrm{T}^{3+2l+v-2s} \quad . \quad (10)$$

by substitution from (3), (6), (7), and (8). Since H/g is constant by (2) we must have

$$3 + 2l + v - 2s = 0,$$

so that

$$s = l + \tfrac{1}{2}(3 + v) \quad . \quad . \quad . \quad . \quad (11)$$

and (10) then becomes

$$\frac{L}{4\pi cGM} = \frac{8}{3(5+v)} \frac{a\mu_0{}^2}{\kappa\rho_0{}^2\mathfrak{R}} \quad . \quad . \quad . \quad (12)$$

Reverting from μ_0, ρ_0 to μ, ρ this becomes

$$\rho^2 = \frac{8}{3(5+v)} \frac{4\pi cGM}{L\kappa} \frac{a}{\mathfrak{R}} \mu^2 T^{3+v} \quad . \quad . \quad . \quad (13)$$

Further, by (9) and (11),

$$T = \frac{2(1-l)\mu\phi}{5+v} \frac{}{\mathfrak{R}} \quad . \quad . \quad . \quad . \quad (14)$$

From (1) and (14) the temperature at any point in the envelope is found, and the density is then found from (13).

If we postulate an extended envelope in a given star, we have to reduce the density in the region occupied by the envelope below that of the standard model $n = 3$. It is therefore useful to give for comparison with (13) and (14) the corresponding formulæ for the model $n = 3$. These are

$$\rho^2 = \frac{1}{3} \frac{4\pi cGM}{L\kappa} \frac{a}{\mathfrak{R}} \mu^2 T^6 \left(\frac{T_c{}^{\frac{1}{2}}}{2{\cdot}5}\right) \quad . \quad . \quad . \quad (13a)$$

$$T = \frac{1}{4} \frac{\mu\phi}{\mathfrak{R}} \quad . \quad . \quad . \quad . \quad (14a)$$

In (13*a*) the denominator 2·5 is an averaging factor (whose precise value depends on the supposed law of distribution of energy), so that $2{\cdot}5T_c{}^{-\frac{1}{2}}$ represents the adopted average value of $\eta T^{-\frac{1}{2}}$ used in my comparisons with observation.*

The explicit differences between the two sets of formulæ are comparatively trivial and we shall ignore them for the moment. The main difference arises from the fact that ϕ has a different value in the two models. In the polytrope ϕ reaches a maximum value $3{\cdot}4GM/R$ at the centre of the star. In the extended envelope ϕ increases more rapidly in accordance with (1) and reaches $4GM/R$ at $r = \frac{1}{5}R$, and $9GM/R$ at $r = \frac{1}{10}R$. Milne's theory requires a very exaggerated envelope, which will at any rate include $r = \frac{1}{5}R$.

Effectively, then, the potentials referring to the middle of the star in the ordinary model become shifted into the envelope, and (ignoring the minor differences) the formulæ show that the corresponding

* Our ultimate aim is to compare the value of κ necessary for the extended envelope with the value of κ from atomic physics. This is done directly in the numerical calculation in § 4 ; but here it is most convenient to use my calculation of κ for the polytropic model as intermediary.

temperatures and densities are shifted along with the potentials. The shift of temperature is inevitable on account of (14) and (14*a*); but the shift of density has got to be counteracted, since it would obviously bring mass into the envelope instead of draining it away. Surveying (13), the only factor left at our disposal is κ. Hence κ must be largely increased in order to bring down the mass of the envelope to the required value. We conclude that the greater the extension of the envelope the greater must be the intrinsic opacity of the material.

Turning to the other differences, (13) and (14) differ from (13*a*) and (14*a*) by a numerical factor $8/(5+v)$. For $v = 3\frac{1}{2}$ this factor is $16/17$, so that it scarcely requires further notice. Since the molecular weight increases as the temperature decreases, l is negative, and the l term slightly increases the temperature and therefore the density in the extended atmosphere. Further, in using the model $n = 3$, we adopt an average molecular weight (usually 2·2) appropriate to the main part of the interior. The envelope will include regions of somewhat higher μ; by (13) and (14) this increases the temperature and density. Finally (taking $v = 3\frac{1}{2}$) the odd $T^{\frac{1}{2}}$ in (13) replaces $T_c^{\frac{1}{2}}/2{\cdot}5$ in (13*a*); this increases the density in the envelope except in the outermost regions where the temperature is below $\frac{1}{6}T_c$. In the main these minor differences tend to increase the density in the envelope so that a further increase of κ is required to counteract them.

4. We shall henceforth omit the l term. As already shown, the effect of this omission is to reduce slightly the temperature and density so that our results will be rather too favourable to the hypothesis of extended envelopes.

Combining (1), (13), and (14) and setting $v = 3{\cdot}5$, we have

$$\rho = \left(\frac{16}{17}\frac{4\pi c\mathrm{GM}}{\mathrm{L}\kappa}\frac{a\mu^2}{3\mathfrak{R}}\right)^{\frac{1}{2}}\left(\frac{4}{17}\frac{\mu}{\mathfrak{R}}\frac{\mathrm{GM}}{\mathrm{R}}\right)^{3\cdot25}\left(\frac{1}{u}-1\right)^{3\cdot25} \quad . \quad (15)$$

where $u = r/\mathrm{R}$.

If the lower limit of the envelope is at $u = \mathrm{U}$, the total mass of the envelope is

$$4\pi\int\rho r^2dr = 4\pi\mathrm{R}^3\int_{\mathrm{U}}^{1}\rho u^2du.$$

By definition of the envelope this is equal to $\frac{1}{10}\mathrm{M}$. Hence we have

$$\frac{1}{10} = \frac{4\pi\mathrm{R}^3}{\mathrm{M}}\left(\frac{16}{17}\frac{4\pi c\mathrm{GM}}{\mathrm{L}\kappa}\frac{a\mu^2}{3\mathfrak{R}}\right)^{\frac{1}{2}}\left(\frac{4}{17}\frac{\mu}{\mathfrak{R}}\frac{\mathrm{GM}}{\mathrm{R}}\right)^{3\cdot25}f(3{\cdot}25,\ \mathrm{U}) \quad . \quad (16)$$

where

$$f(z,\ \mathrm{U}) = \int_{\mathrm{U}}^{1}(1/u-1)^z u^2du \quad . \quad . \quad . \quad (17)$$

The following table gives the values of $f(z,\ \mathrm{U})$ for $\mathrm{U} = \frac{1}{5}$:—

$z=$	2.	3.	4.	5.	6.	7.	8.
$f(z, \frac{1}{5})=$	·171	·319	·774	2·163	6·596	21·225	70·91

For an example I calculate the intrinsic opacity κ of the envelope of a star resembling Krueger 60 on the hypothesis that the radius of the core is $\frac{1}{5}$ of the observed radius, so that $U = \frac{1}{5}$. The adopted data are

$$M = 0{\cdot}25\odot, \qquad L = 0{\cdot}01\odot, \qquad T_e = 3000^\circ, \qquad \mu = 2{\cdot}2.$$

From L and T_e we find

$$R = 2{\cdot}55 \,.\, 10^{10} \text{ cm.}$$

Also (by interpolation in the above table)

$$f(3{\cdot}25, \tfrac{1}{5}) = 0{\cdot}39.$$

We can now calculate κ immediately from (16). The result is

$$\kappa = 1{\cdot}02 \,.\, 10^{29}.$$

My calculation of the astronomical opacity from Capella * gave $\kappa/\mu = 8{\cdot}98 \,.\, 10^{26}$, or $\kappa = 2 \,.\, 10^{27}$. Thus the required opacity is 50 times greater than my astronomical value and therefore at least 500 times greater than the value expected from physical theory. If we had allowed for the fact that in these small stars the gas is slightly superperfect owing to the electrical forces between the ions, the required value of κ would have been still greater.

The temperature in the envelope is by (14)

$$T = \frac{4}{17}\frac{\mu GM}{\mathfrak{R}R}\left(\frac{1}{u} - 1\right)^{\frac{1}{4}}$$

$$= 8{,}140{,}000(1/u - 1).$$

Thus it reaches 32 million degrees at the base of the envelope where $u = \frac{1}{5}$, and only a small region beyond $u = 0{\cdot}89$ is below a million degrees. There is no excuse for a deviation from the expected value of the opacity through the complication of low temperature.

It remains to verify that radiation-pressure in this star is actually small compared with gas-pressure. We have

$$\frac{p_G}{p_R} = \frac{3\mathfrak{R}}{\mu a}\frac{\rho}{T^3} = \frac{3\mathfrak{R}}{\mu a}\left(\frac{16}{17}\frac{4\pi cGM}{L\kappa}\frac{a\mu^2}{3\mathfrak{R}}\right)^{\frac{1}{4}}\left(\frac{4}{17}\frac{\mu}{\mathfrak{R}}\frac{GM}{R}\right)^{\frac{1}{4}}\left(\frac{1}{u} - 1\right)^{\frac{1}{4}} \qquad (18)$$

$$= 16{\cdot}8(1/u - 1)^{\frac{1}{4}}$$

Between $u = 0{\cdot}2$ and $0{\cdot}9$ this ranges from 24 to 10, and even at $u = 0{\cdot}98$, gas-pressure is still more than 6 times radiation-pressure. Thus the star is one in which neglect of radiation-pressure is justified.

5. *Stars with High Radiation-Pressure.*—It is useful to consider the case when gas-pressure is negligible compared with radiation-pressure, since it gives the limit opposite to that which we have just investigated, but it is unlikely that there are any actual stars to which this approximation is at all closely applicable.

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* *Internal Constitution of the Stars*, p. 146.

When the material is wholly supported by radiation-pressure, we have

$$k = 4\pi c\mathrm{GM}_r/\mathrm{L}_r,$$

so that approximately in the envelope

$$k = 4\pi c\mathrm{GM}/\mathrm{L} \quad . \quad . \quad . \quad . \quad (19)$$

Hence by (3)

$$\rho = \frac{4\pi c\mathrm{GM}}{\mathrm{L}\kappa}\mu\mathrm{T}^v \quad . \quad . \quad . \quad . \quad (20)$$

Also, since P is now wholly radiation-pressure,

$$d\mathrm{P} = \tfrac{4}{3}a\mathrm{T}^3 d\mathrm{T}.$$

Hence

$$d\phi = d\mathrm{P}/\rho = \frac{a\mathrm{L}\kappa}{3\pi c\mathrm{GM}\mu}\mathrm{T}^{3-v}d\mathrm{T}$$

and

$$\phi = \frac{a\mathrm{L}\kappa}{(4-v)3\pi c\mathrm{GM}\mu}\mathrm{T}^{4-v} \quad . \quad . \quad . \quad (21)$$

The equations (1), (21), (20) determine successively ϕ, T, ρ, when κ is known. Or we can proceed, as in § 4, to calculate the κ required for any given extension of the envelope by using the integral $f(z, \mathrm{U})$.

By (20) and (21)

$$\rho \propto \phi^{v/(4-v)} \quad . \quad . \quad . \quad . \quad (22)$$

For $v = 3\frac{1}{2}$ this gives $\rho \propto \phi^7$ as compared with $\rho \propto \phi^{3\cdot25}$ in the previous case when the material was wholly supported by gas-pressure. The exponent is sensitive to small changes of v; thus if $v = 3\frac{3}{4}$, $\rho \propto \phi^{15}$. Since the physical formula on which the value $v = 3\frac{1}{2}$ is based does not pretend to any great accuracy, we cannot lay much stress on the exponent 7, but it is clear that when radiation-pressure predominates, ρ varies as a considerably higher power of ϕ than when gas-pressure predominates.

6. *Stars with Intermediate Radiation-Pressure.*—In the ordinary giant stars neither radiation-pressure nor gas-pressure can be neglected, and the direct treatment adopted in the preceding sections is no longer possible. But it is these stars which are supposed to have extended atmospheres in Jeans's theory, and also on the hypothesis which I have proposed for discussion in § 1; so that this is an important case to consider.

We can attain our main object by proceeding in a converse way. We shall assume that the density in the envelope is distributed according to a law

$$\rho = \alpha\phi^s \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and determine the intrinsic opacity at the different points in the envelope necessary to support this distribution. The results will give a law of opacity, which will no longer be of the simple form $k \propto \rho/\mu\mathrm{T}^v$. Adopting different values of s in succession we shall obtain different

laws of variation of opacity, and cover all the cases which we desire to investigate.

We have seen that, for $v = 3\frac{1}{2}$, stars with low radiation-pressure give $s = 3\frac{1}{4}$, and those with high radiation-pressure give $s = 7$; hence we may expect that for intermediate stars, such as Capella, a value of s about 5 will give a law of opacity most nearly agreeing with $v = 3\frac{1}{2}$. For this reason we try values of s in the neighbourhood.

The mass of the envelope is

$$\tfrac{1}{10}\mathrm{M} = 4\pi\int\rho r^2 dr = 4\pi\mathrm{R}^3\alpha(\mathrm{GM/R})^s\int_{\mathrm{U}}^{1}(1/u-1)^s u^2 du,$$

so that

$$\frac{1}{10\alpha} = \frac{4\pi\mathrm{R}^3}{\mathrm{M}}\left(\frac{\mathrm{GM}}{\mathrm{R}}\right)^s f(s,\mathrm{U}) \quad . \quad . \quad . \quad (24)$$

which determines α when U is given.

Also

$$d\mathrm{P} = \rho d\phi = \alpha\phi^s d\phi,$$

so that

$$\frac{\alpha\phi^{s+1}}{s+1} = \mathrm{P} = \frac{\mathfrak{R}}{\mu}\rho\mathrm{T} + \frac{1}{3}a\mathrm{T}^4 = \frac{\mathfrak{R}}{\mu}\alpha\phi^s\mathrm{T} + \frac{1}{3}a\mathrm{T}^4 \quad . \quad (25)$$

We have to solve this equation to determine T for a number of values of ϕ.

Differentiating (25) with respect to ϕ, we have

$$\rho = \frac{\mathfrak{R}\mathrm{T}}{\mu}\frac{d\rho}{d\phi} + \left(\frac{\mathfrak{R}\rho}{\mu} + \frac{4}{3}a\mathrm{T}^3\right)\frac{d\mathrm{T}}{d\phi} \quad . \quad . \quad . \quad (26)$$

But by (10) we have in the envelope

$$\frac{\mathrm{L}}{4\pi c\mathrm{GM}} = \frac{4}{3}\frac{a\mathrm{T}^3}{k\rho}\frac{d\mathrm{T}}{d\phi} \quad . \quad . \quad . \quad (27)$$

Eliminating $d\mathrm{T}/d\phi$ between (26) and (27) we obtain, after a little reduction,

$$k = \frac{4\pi c\mathrm{GM}}{\mathrm{L}}\left(1 - \frac{\mathfrak{R}\mathrm{T}}{\mu}\frac{s}{\phi}\right)\Big/\left(1 + \frac{3}{4}\frac{\mathfrak{R}\rho}{\mu a\mathrm{T}^3}\right) \quad . \quad . \quad (28)$$

so that k can now be found for various values of ϕ. The most convenient way of exhibiting the results is to give the intrinsic opacity κ, determined from k by (3) with $v = 3\frac{1}{2}$, although κ will no longer be constant throughout the envelope.

7. As an example I shall determine the opacity necessary to give Capella an observed radius 5 times the core radius, so that $\mathrm{U} = \frac{1}{5}$. The adopted data are

$$\mathrm{M} = 8{\cdot}30\,.\,10^{33}, \qquad \mathrm{L} = 4{\cdot}80\,.\,10^{35}, \qquad \mathrm{R} = 9{\cdot}55\,.\,10^{11}, \qquad \mu = 2{\cdot}2.$$

We take first $s = 5$; then by (24) we find

$$\alpha = 5{\cdot}39\,.\,10^{-79}.$$

The following results are then obtained from (25) and (28) :—

Capella ($U=\frac{1}{6}$, $s=5$).

u.	ρ.	$T\times10^{-6}$.	k.	$\kappa\times10^{-27}$.
0·2	0·0359	6·63	138·1	6·35
0·333	0·00112	2·56	203·7	10·67
0·5	$3·51\,.\,10^{-5}$	0·955	261·0	13·97
0·8	$3·42\,.\,10^{-8}$	0·127	341·6	16·02

Similarly, for $s = 4$, we find

Capella ($U=\frac{1}{6}$, $s=4$).

u.	ρ.	$T\times10^{-6}$.	k.	$\kappa\times10^{-27}$.
0·2	0·0251	6·74	187·7	13·10
0·333	0·00157	2·94	218·8	13·37
0·5	$9·80\,.\,10^{-5}$	1·27	247·4	12·89
0·8	$3·83\,.\,10^{-7}$	0·234	296·0	10·64

It will be seen that the case $s = 4$ gives nearly constant κ, so that it corresponds to the law $k \propto T^{-3·5}$ usually adopted. But κ has about 6 times the value ($2·10^{27}$) found in my investigation of Capella, according to the ordinary model, and therefore at least 60 times the value expected from physical theory.

Our result is therefore that, in order to give Capella an extension of 5, we require material of 6 times greater intrinsic opacity than is needed in the ordinary model. The extension here considered is a very moderate one ; it is, for example, less than the extension 9·5 required according to the hypothesis suggested in § 1.

The case $s = 5$ is an example of a star with intrinsic opacity increasing outwards ; or, what comes to the same thing, a law of opacity in which k varies faster than $T^{-3·5}$. As we should expect, the same extension is obtained with a lower intrinsic opacity in the inner part of the envelope, this being compensated by the higher intrinsic opacity in the outer part. To show a more extreme case of shifting the high opacity to the outer part of the envelope I have worked out the results for $s = 7$.

Capella ($U=\frac{1}{6}$, $s=7$).

u.	ρ.	$T\times10^{-6}$.	k.	$\kappa\times10^{-27}$.
0·2	0·0586	6·10	68·5	1·44
0·333	$4·57\,.\,10^{-4}$	1·90	198·1	9·08
0·5	$3·57\,.\,10^{-6}$	0·522	303·2	19·14
0·8	$2·18\,.\,10^{-10}$	0·0346	398·9	31·10

Here κ at the inner boundary of the envelope is no larger than in the polytropic model, but it soon increases and becomes very large in the

outer parts of the envelope. A nearly equivalent condition is obtained by treating say $\frac{1}{20}$, instead of $\frac{1}{10}$, of the mass of the star as the envelope. The extension of the star is then to be obtained by inflating a smaller part of the mass, and, as would be expected, a greater intrinsic opacity of this part of the mass is required.

For comparison I have calculated by the same method the opacity necessary to give Capella normal extension $U = \frac{1}{2}$.

Capella ($U=\frac{1}{2}$, $s=4$).

u.	ρ.	$T\times10^{-6}$.	k.	$\kappa\times10^{-27}$.
0·5	$3{\cdot}93\cdot10^{-3}$	2·46	81·5	1·06
0·7	$1{\cdot}33\cdot10^{-4}$	0·933	118	1·54
0·9	$5{\cdot}99\cdot10^{-7}$	0·191	183	2·05

Thus to obtain uniform κ it would be necessary to take s somewhat less than 4. From our experience in the case $U = \frac{1}{5}$, we judge that the uniform value of $\kappa \times 10^{-27}$ will be about 1·5—that is to say, about $\frac{3}{4}$ of the ordinary astronomical value.

Evidently, to accord with the low value of the opacity given by the present physical theory the extension would need to be very much less than normal—that is to say, we should require a deflated envelope.

8. I now turn to consider the suggestion in § 1 that a giant star is practically the same as a main series star of the same mass, except that its envelope is extended so as to give the greater diameter observed. It is difficult to arrive at any definite conclusion without more knowledge of the nature of stellar opacity. There is in any case a rather large discordance between the physical and astronomical determinations of opacity, and everything depends on how big an increase of this discordance we are ready to tolerate.

We have seen that a 6-fold increase of opacity is needed to give Capella an extension of 5; presumably at least a 10-fold increase is needed to give it an extension of $9\frac{1}{2}$, so as to leave a core like that of a main series star of the same mass (Algol). Hence the factor which determines whether a star of mass 4·3 will be like Capella or like Algol is that in the former case the intrinsic opacity of the material of the envelope is 10 times greater than in the latter. The difference seems far too large to be regarded as an accidental difference of constitution. If the opacity of the main interior showed the same degree of variability it would play havoc with any regular mass-luminosity relation.

This test of the sensitivity of the extension to small changes of opacity turns out decidedly unfavourable to our hypothesis. If the hypothesis were right we should expect a great increase of sensitivity at the mass at which giant stars begin to appear. It is true that the change of opacity needed to bring about a given extension in Capella is very much less than in Krueger 60 (§ 4); but there is still no high sensitivity, and there is nothing like an indeterminacy in the extension of the envelope. There would be yet greater difficulty in applying the same hypothesis to the red giants.

Thus far we have been considering the opacity of the envelope. The hypothesis also requires some change (though not a very important one) in the opacity of the inner core. The state of extension of the envelope makes practically no difference to the mechanical and thermal equilibrium of the core, and my theory of the mass-luminosity relation applies unchanged except that the stellar radius to be used in the formulæ is that of a star completed on the polytropic model (*i.e.* the corresponding main series star) and not the extended radius. Thus the luminosity of Capella should be calculated as though its radius were 2,000,000 instead of 9,500,000 km. This change increases the calculated brightness in the ratio $\sqrt{(9{\cdot}5/2)}$ or 2·2. To compensate it we must increase the opacity of the material of the core in the ratio 2·2 ; thus the discordance from the physical prediction is increased in the interior as well as in the envelope. Since this correction applies only to giant stars, it affects differential comparisons, *e.g.* between Capella and the Sun ; but I find that on the whole it does no great harm to the agreement between theory and observation in regard to the mass-luminosity curve.

The only other evidence on the subject is derived from Cepheid variables. Since the typical Cepheids are giant stars they should, on the present hypothesis, have a strong condensation of mass to the centre. But it has been found by J. C. P. Miller * that there is better agreement between the observed and calculated periods when a model rather *less* condensed than the usual polytrope is adopted.

9. *Summary.*—The main result is to show that the outstanding discrepancy between astronomical and physical calculations of stellar opacity is not removed but is made worse if we adopt Jeans's or Milne's theories. Both these authors postulate that part of the mass forms a core of high density which is therefore small in comparison with the volume of the star. As the core is (or is assumed to be) "liquid," a high opacity in this region is not required. But it is shown in this paper that high opacity is required in the tenuous envelope (which fills up the remainder of the observed volume of the star) in order that it may be supported in equilibrium. These theories therefore merely transfer the discordance from the core to the envelope.

Apart from the bearing of this result on the merits of rival theories, it is interesting as showing how persistent is the astronomical demand for high opacity.

In addition, I discuss (on the basis of my own theory) the hypothesis that giant stars are essentially similar to main series stars of the same mass, the inflation being limited to the outer layers—say, the outermost $\frac{1}{10}$ of the mass. Whilst the hypothesis has some attraction as simplifying the scheme of evolution, the detailed investigation is by no means favourable to it. In particular, the required differences of opacity (corresponding to the different extensions in main series stars and in the various types of giants) are found to be large, and it seems unlikely that the material of the envelope could be so variable.

* *M.N.*, **90**, 59, 1929.

Convection in Stars. By Harold Jeffreys, D.Sc., F.R.S.

It is well known that when a mobile fluid is heated below, instability develops when the vertical temperature gradient exceeds the adiabatic rate by a small amount; this is given by an equation of the form

$$\lambda = -g\alpha\beta h^4/\kappa\nu,$$

where λ is a numerical constant depending on the boundary conditions, but always of order 1000, α is the coefficient of expansion, β the excess of the actual temperature gradient over the adiabatic, h the depth, κ the thermometric conductivity, and ν the kinematic viscosity.* It has been shown by Jeans † that radiative transfer of heat and momentum in stellar conditions far exceeds the corresponding mechanical effects. As an illustration of the magnitudes involved we may take

$g = 10^4$ cm./sec.^{2}, $\alpha = 10^{-6}/1°$ C., $h = 10^{11}$ cm., $\kappa = \nu = 16$ cm.^{2}/sec.

Then for instability we must have

$$\beta = -2{\cdot}5 \times 10^{-37} \text{ degrees per cm.},$$

and the difference of temperature between the lower layers of the star and the surface, apart from the adiabatic difference, would be of order $2{\cdot}5 \times 10^{-26}$ degrees.

I am not in a position to express an opinion on the astrophysical questions involved, but it seems to me that some reference to the ease of initiating convection in masses of stellar dimensions must be relevant to modern theories of stellar constitution. If energy is developed in a stellar interior at a low rate, so that it can be disposed of without the adiabatic gradient of temperature being exceeded, a theory of steady flow of heat under hydrostatic conditions may apply. But if the supply is rapid enough to make the temperature gradient exceed the adiabatic by an insignificant amount, the spherically symmetrical distribution becomes unstable, and rising and sinking currents are generated. These will attain such velocities as to redistribute the energy by convection, and the ultimate result will probably be to restore, on the average, the adiabatic gradient for the material in question, together with its associated radiation.

I understand that in current theories of stellar constitution, in which energy is generated throughout the mass, the radiative and conductive transfer disposes of the new heat without the adiabatic gradient being reached. If, on the other hand, the generation of new energy is confined to a small central core this may not be true, and convection may be important. Professor E. A. Milne calls my attention also to the fact that in the ordinary conditions of stellar atmospheres the adiabatic

* Jeffreys, *Proc. Roy. Soc.*, A, **118**, 195–208, 1927; *Proc. Camb. Phil. Soc.*, **26**, 170–172, 1930.

† *M.N.R.A.S.*, **86**, 328–335, 1926.

gradient is not reached, but suggests that convection probably occurs in the umbræ of sunspots, where the gradient is much steeper.

Note added in proof.—Since this paper was communicated, Sir Gilbert Walker's paper on unstable layers in the sun has appeared.* It concerns the polygonal distribution of the currents in a shallow layer when instability has just developed. It may be mentioned that the conversion of the polygons into long strips when there is a shearing current was noticed experimentally by A. R. Low and explained by me in the first paper quoted above.

The Slow Contraction or Expansion of a Fluid Sphere. I. Equilibrium.

By L. H. Thomas, M.A., Ph.D.

1. *Introduction.*—It would seem to be generally assumed † that a sphere composed of perfect gas and contracting as it radiates away its energy will become hotter "in accordance with Lane's law." All that is actually proved, however, is that corresponding to any possible state of dynamical equilibrium of such a sphere there is a series of "homologous" states in which the dimensions are varied in a constant ratio and the temperature in inverse ratio. The accepted argument seems to be that the equilibrium states possible for given energy are nearly alike, and that a good approximation to any must be that one of such a "homologous series" corresponding to the energy at any time. There is, however, no reason at all *a priori* why a contracting sphere should pass through a homologous series, or even why it should continue to contract slowly and steadily and not start collapsing, exploding, or oscillating. The sphere might also expand slowly and steadily instead of contracting. The actual motion can only be predicted from a closer examination of the equations.

2. *The Equations of Motion.*—The equations for the purely radial motion of a fluid sphere that will be used in this paper are:

$$\rho\frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial}{\partial r}(p + \tfrac{1}{3}a\mathrm{T}^4) - \frac{\gamma\rho}{r^2}4\pi\int_0^r \rho r^2 dr \quad . \quad . \quad (1.1)$$

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{\mathrm{D}}{\mathrm{D}t}a\mathrm{T}^4 - (p + \tfrac{4}{3}a\mathrm{T}^4)\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2 c}{3\kappa\rho}\frac{\partial}{\partial r}a\mathrm{T}^4\right) \quad . \quad (1.2)$$

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v) \quad . \quad . \quad . \quad . \quad (1.3)$$

where ρ is the density, p the pressure, T the temperature, u the internal energy per unit mass, κ the opacity, and v the outward radial velocity of

* *M.N.R.A.S.*, **90**, 788, 1930.

† Not, however, by everybody. This paper has been published at the suggestion of Professor E. A. Milne, who has independently arrived at similar general conclusions.

the matter at distance r, $\mathrm{D}/\mathrm{D}t$ being the usual hydrodynamical operator, here $\partial/\partial t + v\partial/\partial r$.

These equations are just those used by Jeans and Vogt,* except that the term representing internal generation has been omitted. In them only terms of order v^2/c^2 of terms which are retained and terms representing radiative viscosity have been omitted. Milne † has replaced equation (1.2) by one differing by

$$-v\frac{\partial}{\partial r}a\mathrm{T}^4 + \frac{a\mathrm{T}^4}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t},$$

but I think that this is incorrect and that some of the results at which he arrives are wrong. A derivation, which I believe to be correct, of the equations, exact to all orders in v^2/c^2, will appear shortly.‡

The radiation pressure and energy per unit mass enter the above equations in such a way that they can simply be added to the material pressure and energy, giving total pressure and energy per unit mass P, U, when the equations take the form §

$$\rho\frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial \mathrm{P}}{\partial r} - \frac{\gamma\rho}{r^2}4\pi\int_0^r \rho r^2 dr \quad . \quad . \quad . \quad (2.1)$$

$$\rho\frac{\mathrm{D}\mathrm{U}}{\mathrm{D}t} - \frac{\mathrm{P}}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2 c}{3\kappa\rho}\frac{\partial}{\partial r}a\mathrm{T}^4\right) \quad . \quad . \quad (2.2)$$

(1.3) being unaltered.

In general U can be expressed in terms of the entropy and volume per unit mass, s and σ $(= 1/\rho)$, and then

$$\frac{\partial \mathrm{U}}{\partial s} = \mathrm{T}, \qquad \frac{\partial \mathrm{U}}{\partial \sigma} = -\mathrm{P} \quad . \quad . \quad . \quad (3)$$

In the particular case of a perfect gas with

$$u = \int \mathrm{C}_v d\mathrm{T}, \qquad p = \mathrm{RT}/\sigma$$

so that

$$\mathrm{U} = a\mathrm{T}^4\sigma + \int \mathrm{C}_v d\mathrm{T}, \qquad \mathrm{P} = \tfrac{1}{3}a\mathrm{T}^4 + \mathrm{RT}/\sigma$$

we have

$$s = \tfrac{4}{3}a\mathrm{T}^3\sigma + \int\frac{\mathrm{C}_v}{\mathrm{T}}d\mathrm{T} + \mathrm{R}\log\sigma \quad . \quad . \quad . \quad (4)$$

the entropy for the radiation and the gas being additive. This gives the exact integral relation between T and σ for adiabatic motion in a convenient form.‖

* *M.N.*, **86**, 328 (1926). Jeans, *Astronomy and Cosmogony*, p. 114.

† *Quarterly Journal of Mathematics* (Oxford), **1**, 1 (1930).

‡ *Quarterly Journal of Mathematics* (Oxford) (1930).

§ This is not possible with Milne's form of the equation (1.2), and the present results disagree with some of his.

‖ The above equations are implicit in Eddington's work (*Internal Constitution of the Stars*, p. 190). See also Milne, *loc. cit.*, p. 14.

The above equations combine to give

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{U}+\tfrac{1}{2}v^2) = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\mathrm{P}v) - \frac{v\gamma\rho}{r^2}4\pi\int_0^r\rho r^2dr + \frac{1}{r^2}\frac{\partial}{\partial r}\Big(\frac{r^2c}{3\kappa\rho}\frac{\partial}{\partial r}a\mathrm{T}^4\Big),$$

an equation which, when multiplied by $4\pi r^2$ and integrated between o and r, gives, using (1.3), the energy change of the fluid inside as

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_0^r\rho(\mathrm{U}+\tfrac{1}{2}v^2)4\pi r^2dr = -4\pi r^2\mathrm{P}v - \int_0^r\Big\{16\pi^2v\gamma\rho\int_0^r\rho r^2dr\Big\}dr + \frac{4\pi r^2c}{3\kappa\rho}\frac{\partial}{\partial r}a\mathrm{T}^4.$$

It is sometimes convenient to change the independent variable from r to m, the mass within distance r of the centre, the operator $\mathrm{D}/\mathrm{D}t$ becoming $\partial/\partial t$; the equations are then

$$\ddot{r} = 4\pi r^2\frac{d}{dm}\frac{\partial\mathrm{U}}{\partial\sigma} - \gamma\frac{m}{r^2} \quad . \quad . \quad . \quad . \quad (5.1)$$

$$\frac{\partial\mathrm{U}}{\partial s}\dot{s} = 16\pi^2\frac{d}{dm}\Big(\frac{r^4c}{3\kappa}\frac{d}{dm}a\Big(\frac{\partial\mathrm{U}}{\partial s}\Big)^4\Big) \quad . \quad . \quad . \quad (5.2)$$

$$\sigma = 4\pi r^2\frac{dr}{dm} \quad . \quad . \quad . \quad . \quad . \quad (5.3)$$

which, if the total mass is M, hold for $\mathrm{o} \leqslant m \leqslant \mathrm{M}$, with boundary conditions $r = \mathrm{o}$ for $m = \mathrm{o}$ and $\partial\mathrm{U}/\partial\sigma = \mathrm{o}$ for $m = \mathrm{M}$.

The first two of these equations can be obtained at once from the third and

$$\delta\int\int_0^{\mathrm{M}}\Big\{\tfrac{1}{2}\dot{r}^2 - \mathrm{U}(s,\sigma) + \gamma\frac{m}{r}\Big\}dmdt = \mathrm{o} \quad . \quad . \quad (6.1)$$

$$\mathrm{T}ds = d\mathrm{Q} \quad . \quad . \quad . \quad . \quad (6.2)$$

3. *The Nature of the Motion.*—The equations last given define the problem very clearly. A conservative dynamical system, defined by (5.3) and (6.1) of a single continuous set of degrees of freedom, like a string, has part of its potential energy, $\mathrm{U}(s,\sigma)$, dependent on a parameter s which is changing very slowly, at any time being a definite function of the independent variable m. The rate of change of s, however, given by (5.2), depends on the state of the system at a given time.

If the initial state is one of equilibrium, *i.e.* if the right-hand side of (5.1) is zero and the boundary conditions are satisfied, three types of initial motion arise.*

(*a*) If the equilibrium is dynamically unstable, the velocities will increase exponentially with a time-period independent of κ.

(*b*) Even if the equilibrium is dynamically stable, the alteration of s at a rate proportional to $1/\kappa$ may cause the velocities to increase

* Sir J. Jeans, *Astronomy and Cosmogony*, pp. 113, 140, but perhaps Jeans's case † No. 1 of "thermodynamical" stability corresponds more nearly to the further motion of a kind mentioned below, or perhaps a slow contraction or expansion would be concluded as unstable.

exponentially with a time-period proportional to $1/\kappa$. In these circumstances the system will, following Jeans, be called " thermodynamically unstable."

(*c*) If it happens that all small velocities—large compared with those necessitated by the change of s—tend to die out, the system will pass slowly through a set of states departing from equilibrium only by very small amounts proportional to $1/\kappa$. It is this type of motion which will be considered below.

In the above discussion no notice has been taken of viscosity ; the radiative viscosity terms are roughly of order v^2/c^2 of the term on the right-hand side of (5.2), and material viscosity is less still. For the small velocities considered, eddy viscosity is not likely to be important.

The above discussion refers only to symmetrical radial motion, but it may well be that equilibrium stable for all symmetrical radial displacements is unstable for other displacements. The condition required for (*c*) is not quite so difficult to satisfy as might appear. However, I leave considerations of stability to a further paper.

4. *The Equations of Equilibrium.*—The succession of equilibrium states through which the sphere may pass are then determined by the equations

$$4\pi r^2 \frac{d}{dm}\frac{\partial \mathrm{U}}{\partial \sigma} - \frac{\gamma m}{r^2} = 0 \quad . \quad . \quad . \quad (7.1)$$

$$\sigma = 4\pi r^2 \frac{dr}{dm} \quad . \quad . \quad . \quad . \quad (7.2)$$

with $r = 0$ for $m = 0$ and $\partial \mathrm{U}/\partial\sigma = 0$ for $m = \mathrm{M}$, and

$$\frac{\partial \mathrm{U}}{\partial s}\dot{s} = 16\pi^2 \frac{d}{dm}\left(\frac{r^4 c}{3\kappa}\frac{d}{dm}a\left(\frac{\partial \mathrm{U}}{\partial s}\right)^4\right) \quad . \quad . \quad (7.3)$$

When s is given as a function of m, (7.1) integrated subject to the boundary conditions leads in general to a unique solution for r and T as functions of m, and (7.3) then enables the change of s, and so the way in which the equilibrium changes, to be found.

5. *The Conditions for Homologous Variation.*—The sphere will be said to contract or expand homologously if the solution of equations (7) is of the form

$$r = f(t) r_0(m) \quad . \quad . \quad . \quad . \quad (8.1)$$

when, by (7.2),

$$\sigma = 4\pi f^3 \frac{dr_0}{dm} r_0^2 \quad . \quad . \quad . \quad . \quad (8.2)$$

and

$$\frac{\partial \mathrm{U}}{\partial \sigma} = \frac{1}{4\pi f^4}\int_0^m \frac{\gamma m}{r_0^4} dm - \frac{\mathrm{P}_0}{f^4} \quad . \quad . \quad . \quad (8.3)$$

where P_0 is an absolute constant given by the boundary condition.

We now have two equations, (8.3), (7.3), restricting the possible forms of three functions of two variables, $\mathrm{U}(s, \sigma)$, $\kappa(s, \sigma)$, $s(m, t)$. Two of these, U and κ, express the physical nature of the fluid ; the third, s,

can be chosen arbitrarily. The equations express a necessary relation between U and κ for any homologous variation to be possible. Any further independent relation between the functions will fix them in terms of arbitrary functions of one variable and of arbitrary constants.

Let us take U to have its value for a perfect gas together with radiation, so that

$$\frac{\partial U}{\partial \sigma} = -P = -\left(\frac{RT}{\sigma} + \frac{1}{3}aT^4\right),$$

and we have by (8.3)

$$\frac{\partial U}{\partial s} = T = \frac{1}{f}T_0(m) \quad \text{(Lane's law)} \qquad . \qquad . \qquad (9)$$

while

$$\frac{\partial U}{\partial s}\dot{s} = T\left[\left(\frac{4}{3}aT^3 + \frac{R}{\sigma}\right)\dot{\sigma} + \left(4aT^2\sigma + \frac{C_v}{T}\right)\dot{T}\right]$$

$$= \frac{f'}{f}(3R - C_v)T.$$

That is, for $C_v < 3R$, f' is negative, and we have contraction; for $C_v > 3R$ there is expansion.

If the specific heat of the gas is independent of the temperature,

$$\frac{\partial U}{\partial s}\dot{s} = \frac{f'}{f^2}(3R - C_v)T_0(m),$$

and by (7.3), using the boundary condition, we have

$$\kappa = \kappa_0(m)(f_0'f^2/f') = \kappa_0(s_0, \sigma_0)(f_0'f^2/f')$$

whence, for homologous variation to be possible,

$$\kappa = T^{-n}\lambda(\rho/T^3) = f^n\kappa_0 \qquad . \qquad . \qquad . \qquad (10)$$

where λ is an arbitrary function and n a constant, and

$$f' = f_0'f^{2-n},$$

or

$$\left.\begin{aligned} f &= b\,(t - t_0)^{-\frac{1}{1-n}} && n < 1 \\ &= b \log (t_0 - t) \quad \text{or} \quad -a \log (t - t_0) && n = 1 \\ &= b\,(t_0 - t)^{\frac{1}{n-1}} && n > 1 \end{aligned}\right\} \qquad . \qquad (11)$$

The formula for the opacity obtained from Kramers's theory,

$$\kappa = 4{\cdot}46 \,.\, 10^{25}\frac{Z^2}{A}\frac{\rho}{\mu T^{3{\cdot}5}} \,*$$

is of the above type with $n = \frac{1}{2}$.

We then have two differential equations of the second order to be satisfied by the two functions $T_0(m)$, $r_0(m)$ which take the form, for

* Jeans, *Astronomy and Cosmogony*, p. 84.

opacity given by (10), when we change back to r as independent variable,

$$\frac{d}{dr}\{\rho RT + \tfrac{1}{3}aT^4\} + \frac{\gamma\rho}{r^2}4\pi\int_0^r \rho r^2 dr = 0 \quad . \quad . \quad (12.1)$$

$$(3R - C_v)\rho T\frac{f'}{f} = \frac{1}{r^2}\frac{d}{dr}\frac{r^2 c}{3\kappa\rho}\frac{d}{dr}aT^4 \quad . \quad . \quad (12.2)$$

with boundary conditions $dT/dr = 0$, for $r = 0$; $T = 0$, for $r = R$ (the last, of course, would only hold to the approximation given by neglecting surface effects).

The solution will involve three arbitrary constants, which can be taken to be the temperature and density at any point, together with the value of f'/f.

It does not, however, follow that if the sphere starts moving along such a homologous series it will continue to do so, even if the motion is dynamically and "thermodynamically" stable. A further condition of stability must also be satisfied, that all small deviations from the homologous variation tend to diminish rather than increase exponentially with the time.

6. *Conclusions.*—We conclude that Lane's law holds for a gaseous sphere contracting homologously, but that in general there is no reason at all why a sphere should contract homologously. In order that homologous contraction should take place it is necessary

(1) that there should be a certain relation between the energy U and opacity κ, which in fact holds for a perfect gas and Kramer's opacity formula,

(2) that the initial distribution of mass and temperature should satisfy a further differential equation besides that of dynamical equilibrium, and

(3) that the state should satisfy a further stability condition as well as those for dynamical and "thermodynamical" stability.

Even if there is such a stable state, it does not follow that the motion from any initial distribution of mass and temperature satisfying the equation of dynamical equilibrium approaches it. Nor, if the sphere is not contracting homologously, does it follow that Lane's law holds, even approximately.

The problem treated above is that of a sphere in which there is no internal generation of energy. The existence of internal generation of energy would make the analysis more complicated, and further stability conditions might have to be introduced. It is very unlikely that Lane's law would hold any more generally.

7. *Summary.*—The equilibrium theory of slow contraction or expansion of a fluid sphere is considered in detail in the case in which there is no internal generation of energy. The conclusion is drawn that the change will not in general be to a homologous density distribution, even if the sphere is gaseous, and that being so, Lane's law cannot be expected to hold. Consideration of the conditions of stability is left to a later paper.

On the Contraction of the Universe. By W. H. McCrea, Ph.D., Edinburgh University, and G. C. McVittie, Ph.D., Leeds University.

(*Communicated by Sir Arthur Eddington, F.R.S.*)

I. *Introduction.*

A good deal of attention has lately been given to the problem of non-stationary universes in general relativity.*

It appears (Eddington) that our actual universe may probably be considered as expanding from an initial state something like "Einstein's world" towards a limiting state like "de Sitter's world." In particular, Eddington, following Lemaître's analysis, has shown that Einstein's world is unstable. If our universe were originally in such a state, any small disturbance would start it expanding or contracting. He suggests that such an initial disturbance is to be found in the concentration of matter, originally in a uniform distribution, into local condensations.

In the present paper we study non-stationary universes in which there is a single condensation of matter at the origin. It is true that in the actual universe the case is somewhat different, for there we have a large number of condensations scattered at random throughout the universe. But the known difficulty of treating non-homogeneous distributions of matter in relativity theory has prompted us to put on record this preliminary investigation. The result we find is that a universe containing a single condensation would start contracting.

II. *The Line-element.*

The first problem is to determine the line-element of a universe containing matter scattered throughout it and, in addition, a concentration of matter around the origin.

The line-element for a homogeneous distribution of matter found by Lemaître † suggests that we consider a line-element of the form

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - a^2(t)\{\gamma(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} \quad . \quad (1)$$

where $\gamma(r)$ is a function of r alone.

The energy tensor in (1) is given by

$$-8\pi T_{\mu\nu} = G_{\mu\nu} - \tfrac{1}{2}(G - 2\lambda)g_{\mu\nu},$$

where $G_{\mu\nu}$ is the contracted Riemann tensor.

* G. Lemaître, *Ann. Société Scientifique de Bruxelles*, **47**A, 49 (April 1927); H. D. Robertson, *Proc. Nat. Acad. Sci. of U.S.A.*, **15**, 822 (1929); Sir A. S. Eddington, *M.N.R.A.S.*, **90**, 668 (1930).

† G. Lemaître, *loc. cit.*

Calculating the $G_{\mu\nu}$ for (1) we get

$$-8\pi T_4{}^4 = -\frac{3\dot{a}^2}{a^2}\Big/\Big(1-\frac{2m}{r}\Big) - \frac{1}{a^2}\Big\{\frac{\gamma'}{r\gamma^2}+\frac{1}{r^2}-\frac{1}{r^2\gamma}\Big\}+\lambda \quad . \quad . \quad (2)$$

$$-8\pi T_1{}^1 = -\Big(\frac{2\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\Big)\Big/\Big(1-\frac{2m}{r}\Big) - \frac{1}{a^2}\Big\{\frac{1}{r^2}-\frac{1}{r^2\gamma(1-2m/r)}\Big\}+\lambda \quad (3)$$

$$-8\pi T_2{}^2 = -8\pi T_3{}^3 = -\Big(\frac{2\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\Big)\Big/\Big(1-\frac{2m}{r}\Big)$$
$$-\frac{1}{2a^2r\gamma}\Big(\frac{1-m/r}{1-2m/r}\Big)\Big\{\frac{\gamma'}{\gamma}+\frac{2m}{r^2(1-2m/r)}\Big\}+\lambda \quad (4)$$

$$-8\pi T_4{}^1 = \frac{\dot{a}}{a^3\gamma}\frac{2m}{r(r-2m)}, \qquad -8\pi T_1{}^4 = -\frac{\dot{a}}{a}\frac{2m}{(r-2m)^2} \quad . \quad (4a)$$

Here a dash denotes differentiation with respect to r and a dot differentiation with respect to t.

To determine γ we postulate that the pressure at any point in the universe is isotropic. This means that

$$T_1{}^1 = T_2{}^2 = T_3{}^3 \quad . \quad . \quad . \quad . \quad (5)$$

From (3), (4), and (5) we get the following equation for γ :—

$$\frac{1}{r}-\frac{1}{(r-2m)}\frac{1}{\gamma} = \frac{1}{2}\frac{r-m}{r-2m}\Big\{\frac{\gamma'}{\gamma^2}+\frac{2m}{r(r-2m)}\frac{1}{\gamma}\Big\} \quad . \quad (6)$$

Put $\mu = \frac{1}{\gamma}$. Then (6) becomes

$$\frac{d\mu}{dr} - 2\Big\{\frac{1}{r-2m}-\frac{m^2}{r(r-m)(r-2m)}\Big\}\mu = -\frac{2}{r}\frac{r-2m}{r-m}$$

or

$$\frac{d}{dr}\Big[\frac{r\mu}{(r-2m)(r-m)^2}\Big] = -\frac{2}{(r-m)^3}.$$

Hence, integrating

$$\mu = \frac{1}{\gamma} = (1-2m/r)\{1-(r-m)^2/R^2\} \quad . \quad . \quad (7)$$

where R^2 is a constant of integration.

Hence (1) may be written

$$ds^2 = \Big(1-\frac{2m}{r}\Big)dt^2 - a^2(t)\Big\{\frac{dr^2}{(1-2m/r)\{1-(r-m)^2/R^2\}} + r^2(d\theta^2+\sin^2\theta d\phi^2)\Big\} \quad (8)$$

We interpret this line-element as that of a universe with a condensation of radius $2m$ at the origin, and which, away from the origin, is spherical and of radius $a(t)R$. For, when r is large, so that m/r is negligible, we have approximately from (8)

$$ds^2 = dt^2 - a^2(t)\Big\{\frac{dr^2}{(1-r^2/R^2)} + r^2(d\theta^2+\sin^2\theta d\phi^2)\Big\},$$

which is the line-element of an expanding or contracting spherical universe of the type given by Lemaître. Secondly, when r is small, neglecting r/R and m/R and taking $a(t) = constant = 1$, we have approximately

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left\{\frac{dr^2}{1-2m/r} + r^2(d\theta^2 + sin^2\,\theta d\phi^2)\right\},$$

which is the Schwarzschild solution for a particle in an empty universe of zero Gaussian curvature.

III. *The Density and Pressure.*

The density of the matter outside the condensation in the universe whose line-element is (8) is, using (2),

$$8\pi\rho = 8\pi T_4{}^4 = \frac{3\dot{a}^2}{a^2}\Big/\left(1 - \frac{2m}{r}\right) + \frac{1}{a^2}\,\frac{(r-m)(3r-5m)}{r^2R^2} - \lambda \qquad (9)$$

The pressure is given by using (3) or (4). It is

$$8\pi p = \left\{\lambda - \frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{1}{a^2R^2}\right\} - \frac{2m}{r-2m}\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) - \frac{1}{a^2R^2}\left(\frac{m^2}{r^2} - \frac{2m}{r}\right) \qquad (10)$$

Hence the proper density $\rho^\times$ is given by

$$\begin{aligned} 8\pi\rho^\times &= 8\pi(\rho - 3p) \\ &= \left\{-4\lambda + 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{a^2R^2}\right)\right\} - 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\frac{2m}{r-2m} \\ &\qquad + \frac{2}{a^2R^2}\left(\frac{4m^2}{r^2} - \frac{7m}{r}\right) \qquad (11) \end{aligned}$$

IV. *Calculation of Total Proper Mass of the Universe.*

Suppose $a = 1 (\dot{a} = \ddot{a} = 0)$ so that we are studying an equilibrium state of the universe. Suppose also that we are dealing with regions where m/r is small and that we may neglect m^2/r^2 and higher powers.

Write $m/r = \epsilon$.

Our universe differs from a spherical Einstein one of radius R in terms of order ϵ. We therefore write

$$\lambda = (1 + \alpha)/R^2 \qquad . \quad . \quad . \quad . \quad (12)$$

where α is a constant of the same order of magnitude as the mean value of ϵ.

We have, then, from (10), (11), (12) for the pressure p and proper density $\rho^\times$ the values

$$8\pi p = \lambda - \frac{1}{R^2} + \frac{2m}{rR^2} = \frac{(2\epsilon + \alpha)}{R^2} \qquad . \quad . \quad (13)$$

$$8\pi\rho^\times = -4\lambda + \frac{6}{R^2} - \frac{14m}{rR^2} = \frac{\{2 - (14\epsilon + 4\alpha)\}}{R^2} \qquad . \quad (14)$$

Now when $\dot{a} = 0$ we have, by (4a), that $T_4^1 = 0$, so that there is no mass motion of the fluid. The pressure p is therefore caused by a random motion of the molecules of fluid. Let v be the mean random velocity. Then

$$p = \tfrac{1}{3}\rho v^2. \quad . \quad . \quad . \quad . \quad (15)$$

The volume-element of the space is given by

$$dV = r^2 \sin\theta dr d\theta d\phi \Big/ \sqrt{1 - \frac{2m}{r}}\sqrt{1 - \frac{(r-m)^2}{R^2}} \quad . \quad (16)$$

Hence the volume-element appropriate to $\rho^\times$, the *proper* density, for a mean velocity v is $dV/\sqrt{1 - v^2}$, where we take the velocity of light to be unity in the correcting term. We then obtain for the element of proper mass the value, to our previous order of approximation,

$$\begin{aligned} &\rho^\times(1 + \tfrac{1}{2}v^2)dV \\ &= (\rho^\times + \tfrac{3}{2}p)dV \\ &= \{1 - (\tfrac{11}{2}\epsilon + \tfrac{5}{4}\alpha)\}dV/4\pi R^2 \quad . \quad . \quad . \quad (17) \end{aligned}$$

where in employing (15) in the correction term it is not necessary to distinguish between ρ and $\rho^\times$.

The total proper mass in any region of r is given by the integral

$$\frac{1}{R^2}\int\left\{1 - \left(\frac{11}{2}\epsilon + \frac{5}{4}\alpha\right)\right\}\frac{r dr}{\sqrt{1 - 2\epsilon}\sqrt{1 - \frac{(r-m)^2}{R^2}}} \quad . \quad (18)$$

taken through the region. We have integrated for $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$.

After making the substitutions

$$r - m = x, \qquad x = R\sin\chi,$$

the integral (18) becomes, still supposing ϵ small,

$$R\int\left\{\left(1 - \frac{5}{4}\alpha\right)\sin^2\chi - \frac{5}{2}\frac{m}{R}\sin\chi\right\}d\chi \quad . \quad . \quad (19)$$

Now when the sun is represented as a "particle" in a Schwarzschild metric its "m" comes out as 1·5 km., whilst its radius is 7×10^5 km. Working on this analogy we may suppose our "particle" m to be spread out through a volume of radius r_0, large compared with m. So throughout the rest of space ϵ is really small, as we have been assuming. Hence to find the proper mass M in the whole of space outside the particle we should make the lower limit in the integral (19) correspond to r_0, and the upper limit is $\chi = \pi$. But since r_0 must correspond to a very small value of χ, and since the integral in (19) behaves like χ in the neighbourhood of $\chi = 0$, we may take the lower limit to be $\chi = 0$. Performing the integration we find then

$$M = R\left\{\left(1 - \frac{5}{4}\alpha\right)\frac{\pi}{2} - \frac{5m}{R}\right\}.$$

Adding to this the mass, m, of the condensation itself, we get that the total mass of the whole universe, $M^{\times}$, is

$$M^{\times} = M + m = R\left\{\left(1 - \frac{5\alpha}{4}\right)\frac{\pi}{2} - \frac{4m}{R}\right\} \quad . \quad . \quad (20)$$

V. *The Calculation of* λ.

We have calculated the *proper* mass because we may assume that this is conserved throughout any changes undergone by the universe.* Hence if the universe started in an Einstein state it must have been one corresponding to mass $M^{\times}$. But λ is also an absolute constant, so that we must have the relation

$$\frac{2}{\pi}M^{\times} = \frac{1}{\sqrt{\lambda}} \quad . \quad . \quad . \quad . \quad (21)$$

satisfied throughout.

From (12), (20), and (21) we obtain

$$\frac{2}{\pi}M^{\times} = R\left(1 - \frac{1}{2}\alpha\right) = R\left(1 - \frac{5\alpha}{4}\right) - \frac{8m}{\pi},$$

giving

$$\alpha = -\frac{32}{3\pi}\frac{m}{R}$$

$$M^{\times} = \frac{\pi}{2}R\left(1 + \frac{16}{3\pi}\frac{m}{R}\right) \quad . \quad . \quad . \quad (22)$$

VI. *The Contraction of Space.*

The total measured volume of space outside the condensation, whose volume we may neglect in comparison, is

$$V = 4\pi R^3 \int_0^{\pi}\left(sin^2\chi + \frac{3m}{R}sin\,\chi\right)d\chi$$

$$= 2\pi^2 R^3\left(1 + \frac{12}{\pi}\frac{m}{R}\right) \quad . \quad . \quad . \quad . \quad (23)$$

Putting this in terms of $M^{\times}$ from (22) we obtain

$$V = \frac{16}{\pi}M^{\times 3}\left(1 - \frac{2m}{M^{\times}}\right) \quad . \quad . \quad . \quad . \quad (24)$$

This result shows that the equilibrium state of a universe of given total proper mass, in which a condensation has formed, is one of smaller total volume than the equilibrium state for a uniform distribution.† (Einstein universe.)

If, then, we start with an Einstein universe and allow a condensation

* We are neglecting radiation ; *cf.* Eddington, *loc. cit.*

† The result is in agreement with that for an entirely different model of the universe given by W. H. McCrea (in press), *Proc. Edin. Math. Soc.* (2), II., pt. 3.

to form in it we cannot, of course, say that the universe must pass over to the equilibrium state for the non-uniform distribution. But it does seem legitimate to argue from the result just obtained that when the condensation starts to form, the Einstein universe starts contracting. Now Eddington has shown that an Einstein universe is unstable and that if once a contraction or an expansion can be set up in it, the rate of contraction or expansion will continually increase with the time. It seems, therefore, that a condensation starting in an Einstein universe would cause this universe to collapse.

We are aware that this argument is not entirely satisfactory in that we try to show how a system *departs* from equilibrium by studying a series of equilibrium configurations. But an exact treatment of the present problem, viz. : a discussion of general gravitational instability in a Lemaître universe, presents such formidable mathematical difficulties that we have not been able to make any headway in this direction. We therefore have been driven to use the crude method developed in this paper. It would appear, however, that the result is not dependent on the particular model of the universe (represented by the line-element (8)) chosen, but is quite general.

We must conclude that if the actual universe started as an Einstein universe and is now expanding, as the work of Lemaître and Eddington suggests, this expansion cannot have been caused by a redistribution of the matter into massive nuclei. Thus, as yet, no mechanism for setting up the expansion has been found.

In conclusion, we should like to thank Sir Arthur Eddington for many valuable suggestions and criticisms made whilst the work was being carried out.

Note on the Temperature Distribution in the Deep Layers of Stellar Atmospheres. By M. Bronstein.

(*Communicated by Prof. E. A. Milne.*)

It is known that Milne's integral equation

$$B(\tau) = \tfrac{1}{2}\int_0^\infty \mathrm{Ei}(|\tau - t|)B(t)dt$$

has one and only one continuous solution $\phi(\tau)$, different from zero and satisfying the relation $\lim_{\tau\to\infty} \frac{\phi(\tau)}{\tau} = 1$, and that every continuous solution of this equation must be equal to $\phi(\tau)$ multiplied by some numerical factor. We call $\phi(\tau)$ Hopf's function. Let σ be Stefan's constant and T the absolute temperature considered as function of the optical depth τ ; then the function $B(\tau) = \frac{\sigma}{\pi}T^4$ is, as has been shown by

Hopf and by the present writer,* connected to $\phi(\tau)$ by means of the relation

$$B(\tau) = \frac{3\sigma}{4\pi} T_{\text{eff}}^4 \phi(\tau),$$

where T_{eff} is the effective temperature of the star.

Let us denote by $Ei^{(n)}(\tau, t)$ the n-times iterated kernel of Milne's equation :

$$Ei^{(n)}(\tau, t) = \int_0^\infty \int_0^\infty \dots \int_0^\infty Ei(|\tau - x_1|) Ei(|x_1 - x_2|) \dots Ei(|x_{n-1} - t|) dx_1 dx_2 \dots dx_{n-1}.$$

The resolvent $\Gamma(\tau, t) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} Ei^{(n)}(\tau, t)$ satisfies, as can easily be seen, the following equation

$$\Gamma(\tau, t) - \tfrac{1}{2}\int_0^\infty Ei(|\tau - \xi|)\Gamma(\xi, t) d\xi = Ei(|\tau - t|).$$

By putting $t \to \infty$ we obtain

$$\Gamma(\tau, \infty) = \tfrac{1}{2}\int_0^\infty Ei(|\tau - \xi|)\Gamma(\xi, \infty) d\xi.$$

The function $\Gamma(\tau, \infty)$, as we see, satisfies the same integral equation as $\phi(\tau)$. $\Gamma(\tau, \infty)$ is continuous and not identically zero, for it can be easily concluded from a relation † proved by Ambarzumian, viz. :

$$\Gamma(0, s) = 2\sqrt{3}\frac{d}{ds}\phi(s)$$

that

$$\Gamma(0, \infty) = 2\sqrt{3}\left[\frac{d}{ds}\phi(s)\right]_{s=\infty},$$

and this last value is certainly not zero, because it follows from $\lim_{s\to\infty} \frac{\phi(s)}{s} = 1$, by means of elementary theorems, that $\lim_{s\to\infty} \frac{d\phi(s)}{ds} = 1$.

We see that at least one value of $\Gamma(\tau, \infty)$ is not zero (namely, its value when $\tau = 0$, which is $2\sqrt{3}$) ; therefore

$$\Gamma(\tau, \infty) = c\phi(\tau),$$

where c is a numerical coefficient, the value of which can be easily determined. We make $\tau = 0$ and, remembering that $\phi(0) = \frac{1}{\sqrt{3}}$

* E. Hopf, *Monthly Notices*, **90**, 287, 1930 ; M. Bronstein, *Zeits. für Phys.*, **58**, 696, 1929.

† V. Ambarzumian, *Zeits. für Phys.*, **52**, 263, 1928 ; M. Bronstein, *ibid.*, **59**, 144, 1929.

(which result is contained in the above-mentioned papers by Hopf and by the present writer), we obtain $c = 6$ and finally

$$\Gamma(\tau, \infty) = \Gamma(\infty, \tau) = 6\phi(\tau).$$

Now from Hopf's series

$$\phi(\tau) = \tau + \sum_{n=1}^{\infty} \chi_n(\tau),$$

where

$$\chi_1(\tau) = \mathrm{E}i_2(\tau) = \int_1^{\infty} \frac{e^{-\tau\xi}}{\xi^3} d\xi, \qquad \chi_n(\tau) = \tfrac{1}{2}\int_0^{\infty} \mathrm{E}i(|\tau - \xi|)\chi_{n-1}(\xi)d\xi,$$

we can easily deduce

$$\phi(\tau) - \tau = \tfrac{1}{2}\mathrm{E}i_2(\tau) + \tfrac{1}{4}\int_0^{\infty} \Gamma(\tau, t)\mathrm{E}i_2(t)dt.$$

By making $\tau \to \infty$ we obtain

$$\lim_{\tau\to\infty} [\phi(\tau) - \tau] = \tfrac{1}{4}\int_0^{\infty} \Gamma(\infty, t)\mathrm{E}i_2(t)td$$

and finally

$$\lim_{\tau\to\infty} [\phi(\tau) - \tau] = \tfrac{3}{2}\int_0^{\infty} \mathrm{E}i_2(t)\phi(t)dt.$$

This formula can be used for the computation of the numerical value of $\lim_{\tau\to\infty} [\phi(\tau) - \tau]$. The simple methods that made possible the exact evaluation of the integrals $\int_0^{\infty} \mathrm{E}i(t)\phi(t)dt$ and $\int_0^{\infty} \mathrm{E}i_1(t)\phi(t)dt$ fail in the case of $\int_0^{\infty} \mathrm{E}i_2(t)\phi(t)dt$, but the value can be found by successive approximations. In the first approximation, from the known inequality

$$t + \tfrac{1}{2} < \phi(t) < t + 1$$

and from the formula

$$\int_0^{\infty} (t + k)\mathrm{E}i_2(t)dt = \frac{1}{4} + \frac{k}{3}$$

it can be deduced that

$$\tfrac{5}{12} < \int_0^{\infty} \mathrm{E}i_2(t)\phi(t)dt < \tfrac{7}{12},$$

and therefore

$$\tfrac{5}{8} < \lim_{\tau\to\infty} [\phi(\tau) - \tau] < \tfrac{7}{8}.$$

In the second approximation, which can be derived from the first by the known method of Milne,* we must use the inequalities

$$t + \frac{1}{2} - \frac{t}{4}e^{-t} + \frac{t + t^2}{4}\mathrm{E}i(t) < \phi(t) < t + 1 - \frac{t + 1}{4}e^{-t} + \frac{2t + t^2}{4}\mathrm{E}i(t).$$

* E. A. Milne, *Monthly Notices*, **81**, 361, 1921.

We must also make use of the easily deducible formulæ

$$\int_0^\infty e^{-t}\mathrm{E}i_2(t)dt = ln2 - \tfrac{1}{2},$$

$$\int_0^\infty te^{-t}\mathrm{E}i_2(t)dt = 3ln2 - 2,$$

$$\int_0^\infty t\mathrm{E}i(t)\mathrm{E}i_2(t)dt = ln2 - \tfrac{5}{8},$$

$$\int_0^\infty t^2\mathrm{E}i(t)\mathrm{E}i_2(t)dt = \tfrac{14}{5}ln2 - \tfrac{19}{10}.$$

These formulæ allow us to write inequalities

$$\tfrac{137}{480} + \tfrac{1}{5}ln2 < \int_0^\infty \mathrm{E}i_2(t)\phi(t)dt < \tfrac{101}{240} + \tfrac{1}{5}ln2$$

and

$$\tfrac{137}{320} + \tfrac{3}{10}\,ln2 < \lim_{\tau\to\infty}\,[\phi(\tau) - \tau] < \tfrac{101}{160} + \tfrac{3}{10}\,ln2.$$

While the first approximation gave

$$0{\cdot}625 < \lim_{\tau\to\infty}\,[\phi(\tau) - \tau] < 0{\cdot}875,$$

the second now gives

$$0{\cdot}636 < \lim_{\tau\to\infty}\,[\phi(\tau) - \tau] < 0{\cdot}8392.$$

It is not worth while to continue these calculations, which grow more and more complicated with every successive degree of approximation. We can see at once that the value of $\phi(\tau) - \tau$ at infinite τ is not equal to its value at $\tau = 0$, which is $\frac{1}{\sqrt{3}}$ or 0·57735 as has been shown by Hopf and by the present writer. Therefore it is not possible to construct a linear approximation for $\phi(\tau)$, which would be valid at once for very small and very great values of τ.

The knowledge of the value of $\lim_{\tau\to\infty}\,[\phi(\tau) - \tau]$ would be useful for establishing an approximate linear dependence of T^4 on the optical depth, which would be valid in the very deep layers of a star's atmosphere.

The foregoing calculations were made in answer to a question proposed to me by Professor Milne in a private letter.

Physico-Technical Institute, Leningrad:
1930 *September* 13.

Note by Prof. E. A. Milne.—In the foregoing paper Dr. Bronstein refers to his paper "Über das Verhältnis der effektiven Temperatur der Sterne zur Temperatur ihrer Oberfläche" (*Zeits. für Phys.*, **59,** 144, 1929). This paper establishes the same result as was obtained by Dr. Hopf in his paper "Remarks on the Schwarzschild-Milne Model

of the Outer Layers of a Star" (*M.N.*, **90**, 287, 1930), namely, $T_0^4 = \frac{\sqrt{3}}{4} T_{\text{eff}}^4$, the exact form of Schwarzschild's classical approximation, $T_0^4 = \frac{1}{2} T_{\text{eff}}^4$, for the relation between the effective temperature T_{eff} and the boundary temperature T_0 in radiative equilibrium. Dr. Bronstein's paper is dated 1929 Nov. 4; Dr. Hopf's paper is dated 1930 Jan. 1. As I communicated and translated Dr. Hopf's paper, I think it desirable to record that the number of *Zeits. für Phys.* containing Dr. Bronstein's published paper and the private letter containing the German MS. of Dr. Hopf's paper were received by me in the same package of mail on 1930 Jan. 4. This circumstance was mentioned when I gave an account of Dr. Hopf's paper at the January meeting of the Royal Astronomical Society, but it escaped mention in the report of the meeting given in *Observatory*, 1930 February. We have a clear case of the independent and almost simultaneous discovery of an important result by two different investigators, though the actual priority of publication rests with Dr. Bronstein. My account at the January meeting of the Society was based on the original text of Dr. Hopf's paper; the translation was undertaken later at the request of the Council.

Note added by the Author on 1930 *December* 12.—Dr. Ambarzumian has recently told me that on applying a method quite different from that described here, he has obtained the inequality

$$0{\cdot}669 < \lim_{\tau \to \infty} [\phi(\tau) - \tau] < 0{\cdot}780.$$

The lower limit can be obtained by the following method: integrating by parts we find

$$\int_0^\infty \mathrm{E}i_2(t)\phi(t)dt = -\int_0^\infty \phi(t)d\mathrm{E}i_3(t) = \Big[\phi(t)\mathrm{E}i_3(t)\Big]_\infty^0 + \int_0^\infty \mathrm{E}i_3(t)\frac{d}{dt}\phi(t)dt.$$

Here Dr. Ambarzumian makes use of a general relation proved by him, viz.:

$$\mathrm{E}i_n(t) \geqslant \frac{m}{n}\mathrm{E}i_m(t) \quad \text{when} \quad n > m,$$

which gives $\mathrm{E}i_3(t) \geqslant \frac{2}{3}\mathrm{E}i_2(t)$, and therefore

$$\int_0 \mathrm{E}i_2(t)\phi(t)dt > \phi(0)\mathrm{E}i_3(0) + \tfrac{2}{3}\int_0^\infty \mathrm{E}i_2(t)\frac{d}{dt}\phi(t)dt.$$

Integrating by parts again we obtain

$$\int_0^\infty \mathrm{E}i_2(t)\phi(t)dt > \frac{1}{3\sqrt{3}} + \frac{2}{3}\Big[\phi(t)\mathrm{E}i_2(t)\Big]_0^\infty + \frac{2}{3}\int_0^\infty \phi(t)\mathrm{E}i_1(t)dt,$$

and therefore

$$\lim_{\tau \to \infty} [\phi(\tau) - \tau] > \int_0^\infty \mathrm{E}i_1(t)\phi(t)dt.$$

This last integral, evaluated by the present writer, is $\frac{2}{3}$. The exact formula is

$$\lim_{\tau\to\infty}[\phi(\tau)-\tau] = \tfrac{2}{3} + \int_0^\infty [\tfrac{3}{2}\mathrm{E}i_3(t) - \mathrm{E}i_2(t)]\frac{d}{dt}\phi(t)dt.$$

The last integral, as Dr. Ambarzumian proves, is greater than 0·003.

The derivation of the upper limit is more complicated. It is based on the fact that the integral equation

$$\tfrac{1}{3}\lim_{\tau\to\infty}[\phi(\tau)-\tau] + \tfrac{1}{2}[\mathrm{E}i_2(\tau) - \mathrm{E}i_4(\tau)] = y(\tau) - \tfrac{1}{2}\int_0^\infty [\mathrm{E}i(|\tau-t|) - \mathrm{E}i_2(|\tau-t|)]y(t)dt$$

is satisfied by the function $\phi(\tau)-\tau$. From this integral equation and from the inequality

$$\mathrm{E}i_2(\tau) - \mathrm{E}i_4(\tau) \geqslant \tfrac{3}{8}[\mathrm{E}i_1(\tau) - \mathrm{E}i_3(\tau)]$$

follows another inequality

$$\phi(\tau) - \tau > \mathrm{Y}(\tau),$$

where $\mathrm{Y}(\tau)$ is the solution of the integral equation

$$\mathrm{Y}(\tau) - \tfrac{1}{2}\int_0^\infty [\mathrm{E}i(|\tau-t|) - \mathrm{E}i_2(|\tau-t|)]\mathrm{Y}(t)dt = \tfrac{1}{3}\lim_{\tau\to\infty}[\phi(\tau)-\tau] + \tfrac{3}{16}[\mathrm{E}i_1(\tau) - \mathrm{E}i_3(\tau)].$$

But

$$\mathrm{Y}(\tau) > \tfrac{3}{8} + \{\tfrac{1}{3}\lim_{\tau\to\infty}[\phi(\tau)-\tau] - \tfrac{1}{8}\}(1 + \tfrac{1}{3} + \tfrac{1}{9} + \ldots)$$

for it can be proved that the terms of Neumann's series for the function $f(\tau) = \mathrm{Y}(\tau) - \frac{3}{8}$ considered as the solution of the integral equation

$$f(\tau) - \tfrac{1}{2}\int_0^\infty [\mathrm{E}i(|\tau-t|) - \mathrm{E}i_2(|\tau-t|)]f(t)dt = \tfrac{1}{3}\lim_{\tau\to\infty}[\phi(\tau)-\tau] - \tfrac{1}{8}$$

are greater than the corresponding terms of the geometrical series $1 + \frac{1}{3} + \frac{1}{9} + \ldots$ multiplied by $\{\frac{1}{3}\lim_{\tau\to\infty}[\phi(\tau)-\tau] - \frac{1}{8}\}$. It follows that

$$\frac{1}{\sqrt{3}} = \phi(0) > \mathrm{Y}(0) > \tfrac{3}{8} + \tfrac{3}{2}\{\tfrac{1}{3}\lim_{\tau\to\infty}[\phi(\tau)-\tau] - \tfrac{1}{8}\},$$

and finally that

$$\lim_{\tau\to\infty}[\phi(\tau)-\tau] < \frac{2}{\sqrt{3}} - \frac{3}{8} < 0{\cdot}780.$$

I am greatly indebted to Dr. Ambarzumian for his kind permission to give here this brief account of these new results obtained by him.

The Theoretical Contours of Absorption Lines.

By Prof. A. Pannekoek.

1. The problem of determining the contours of absorption lines in the solar spectrum and in stellar spectra has been treated in different papers by Schwarzschild, Milne, Eddington, Unsöld, etc. Schwarzschild and Schuster derived formulæ for the intensity of the radiation emitted from the outer layers of an atmosphere in the case of black absorption and emission as well as in the case of diffusion (scattering). From the visibility of the Fraunhofer lines at the border of the sun Schwarzschild concluded that diffusion plays an important part in the production of these lines. Unsöld * applied his formulæ in the case of pure diffusion to the wings of the broad lines of Ca^+ and H, by making use of Voigt's formulæ for the absorption coefficient; this treatment gives zero for the central intensity, at variance with observation. Milne † showed that the diffusion formulæ are valid in the case of monochromatic equilibrium, whereas the collisions of electrons with the radiating atoms introduce black-body emission and absorption into the formulæ. Because an exact solution presents mathematical difficulties he gives a solution for the case of a constant ratio of the diffusion and the absorption coefficients. This solution forms the basis of his computations concerning ionization effects on line intensities. Eddington's formulæ ‡ are based on the same supposition, but he takes account of the oblique directions in which the radiation traverses the atmosphere. Unsöld § has made use of Milne's solution in deriving results on central intensities; Woolley || has done the same, using Eddington's formulæ.

In these results the residual intensity depends only on the ratio of the diffusion and the black absorption coefficients; their absolute values do not enter into the formulæ. This ratio is assumed to be constant for the whole atmosphere; so the result is obtained that in the centre of a Fraunhofer line, where the general absorption may be neglected, the intensity is the same for the members of a multiplet (Unsöld), and depends on a certain average pressure (Woolley). In reality this ratio is not constant because the absorption coefficients increase with pressure, *i.e.* with the depth in the atmosphere; hence the residual intensity must be smaller for larger absolute values of the coefficients, because the depth from which the radiation reaches the surface decreases with increasing absorption.

Eddington expressed strong doubts whether, through the complications of the problem in its physical as well as its mathematical and astronomical foundations, the results obtained by these approximations so far have any real value. To remove such doubts it will be necessary

* *Zeitschr. für Physik*, **44**, 793 (1927).
† *M.N.*, **88**, 493 (1928).
‡ *Ibid.*, **89**, 620 (1929).
§ *Sommerfeld Festschrift*, p. 95 (1928); *Astroph. J.*, **69**, 277 (1929).
|| *M.N.*, **90**, 170 (1929).

to work out exactly the consequences of strictly defined suppositions ; only when the lack of harmony with observation cannot be ascribed to the approximate mode of treatment, shall we be able to trace its origin back to the physical and astronomical foundations. Therefore an attempt is made in this paper to find an exact solution of the equations representing existing conditions, though simplified, as nearly as possible.

2. The equations are to represent the following conditions and suppositions :—

(*a*) In the atmosphere only two vertical streams of energy are assumed to exist, one going outward, one inward. Schwarzschild, for the case of diffusion, Milne, for the case of absorption, have derived exact formulæ for radiation passing through the atmosphere in every direction. The results deviate only slightly from the simplified case, if only certain numerical factors (1/2, 3/2) are added to the absorption and diffusion coefficients and the optical depth, to take account of the average inclination of the energy streams. A comparison of Eddington's result for the residual intensity with Milne's, if expressed by the same symbols, shows that here the influence of oblique radiation only introduces a coefficient $2/3\sqrt{3} = 1{\cdot}15$ into some of the terms. Therefore these factors will be omitted in our formulæ.

(*b*) We consider one kind of atoms having two energy levels, for which the difference of energy corresponds to the frequency ν_0 ; we consider only frequencies ν slightly different from ν_0, belonging to the realm of the absorption line or to the part of the continuous background adjacent to it. The exchanges of energy to be considered here are : 1*a*, the change of radiation energy of the field into quantum energy through absorption by the atom ; 1*b*, the reverse, change of quantum energy into radiation through emission by the atom ; 2*a*, change of translation energy of electrons into quantum energy by inelastic collisions ; 2*b*, change of quantum energy of the atom into translation energy by hyperelastic collisions. Milne * derived the equation for the radiation of frequency ν ; putting n_1 and n_2 the number of atoms in state 1 and 2 per unit volume, $B_{12}I(\nu)$ the probability of transition $1 \rightarrow 2$ by incident radiation $I(\nu)$, b_{12} the probability of the same transition by electron collisions (for which a Maxwellian velocity distribution with temperature T is assumed), $E(\nu, T)$ the black-body radiation according to Planck's formula, $\eta = b_{12}/B_{12}E(\nu, T)$ the ratio of the number of transitions by electron collisions and by absorption of black radiation of the same temperature T, we have

$$\frac{dI(\nu)}{(n_1 - n_2)B_{12}h\nu dx} = -I + \frac{\int I(\nu)\frac{d\omega}{4\pi} + \eta E(\nu, T)}{1 + \eta}.$$

Substituting macroscopic quantities for the atomic values

$$(n_1 - n_2)B_{12}h\nu = s_0\rho = (k + s)\rho ; \quad \eta = k/s,$$

* "The Effect of Collisions on Monochromatic Radiative Equilibrium," *M.N.*, **88**, 493 (1928).

this equation becomes

$$\frac{dI(\nu)}{\rho dx} = -(k+s)I(\nu) + s\int I(\nu)\frac{d\omega}{4\pi} + kE(\nu, T).$$

Here it appears that the complete monochromatic absorption coefficient s_0 is divided into two parts ; the part s is emitted directly by the atom and works as a coefficient of diffusion ; the part k is transformed into energy of electron translation (acting as black matter) and is returned as black radiation.

(*c*) There is a general absorption and emission for all wave-lengths, which produces the continuous background of the spectrum. We suppose it to be produced by direct transformation of radiation into translation energy, and *vice versa* at electron collisions with atoms of all kinds : $\frac{1}{2}m(v_1^2 - v_2^2) = h\nu$. The coefficient of this general absorption k_0 is assumed to be proportional to $PT^{-3/2}$ (P = electron pressure), according to Gaunt's formula. In the centre of an absorption line it is negligible compared with the line absorption s_0 ; in the wings of course it is important.

(*d*) The probability coefficient of collisions b_{12} is in the same way assumed to be proportional to the electron pressure ; thus we have $\eta = k/s \sim P$.

(*e*) The total absorption in each space element of the atmosphere, which determines the temperature distribution, is composed of this general absorption and the sum total of all line absorptions. After Minnaert the latter may be estimated as 15 per cent. of the total absorption. Introducing a homogeneous depth $d\xi = -\rho dh$, and an optical depth τ by $d\tau = \bar{k}d\xi$, where $\bar{k}$ is the total absorption coefficient $k_0 + k$ for all ν, we have for the total black-body radiation $E = E_0(1 + \tau)$.

(*f*) We assume a constant composition of the atmosphere. Differences of ionization at different depths will not be considered in this paper.

3. We first treat the continuous spectrum background ; for these wave-lengths outside the lines we suppose that only the general absorption with coefficient k_0 is working. For the intensity I and I′ of the two vertical streams of energy we have

$$\frac{dI}{d\xi} = k_0 I - k_0 E\,; \quad \frac{dI'}{d\xi} = -k_0 I' + k_0 E.$$

Introducing $d\tau = \bar{k}d\xi$, and putting $k_0/\bar{k} = n$, not varying with the depth, and further putting $I + I' = y$, $I - I' = z$, we have

$$\frac{dy}{d\tau} = nz, \qquad \frac{dz}{d\tau} = n(y - 2E).$$

By means of the boundary conditions $y_0 = z_0$ for $\tau = 0$, and $y = 2E$ for $\tau = \infty$, we find the radiation leaving at the surface

$$I_0 = y_0 = z_0 = \int_0^\infty E(\tau)e^{-n\tau}nd\tau.$$

This holds for each wave-length λ, frequency ν, outside the absorption lines. The emission function $E(\lambda, \tau)$ depends on τ through the temperature T, for which $T^4 = T_0^4(1 + \tau)$. To a first approximation we may put $E(\lambda, \tau) = E(\lambda, 0)(1 + c\tau)$, where c is written for $\frac{1}{4}h\nu/kT = \frac{1}{4}c_2/\lambda T$. For $T = 6000°$ this coefficient is 1 for $\lambda = 5950$ A.; for λ 4000 it is nearly 1·5. Then the integration gives

$$I_0 = E(\lambda, 0)\left(1 + \frac{c}{n}\right) = E(\lambda, 0)(1 + ck/k_0) \quad . \quad . \quad (1)$$

If we had taken $c = 1$ and neglected the difference between the general and the total absorption the net outward stream of energy of wave-length λ would have been $F(\lambda) = 2E(\lambda, 0)$. The deviation in our formula expresses first the fact, that for shorter wave-lengths, where $c > 1$, the contrast between the continuous spectrum and the absorption lines is stronger than for longer wave-lengths; and, secondly, that this contrast is enhanced by the Fraunhofer lines contributing to the total absorption, whereby the temperature gradient is increased and emission from hotter depths reaches the surface and is radiated.

4. For a wave-length within the absorption line we have both an absorption and a diffusion term. The equations for the two vertical energy streams are now

$$\frac{dI}{d\xi} = (k + s)I - kE - \tfrac{1}{2}s(I + I'),$$

$$\frac{dI'}{d\xi} = -(k + s)I' + kE + \tfrac{1}{2}s(I + I').$$

Again introducing $y = I + I'$, $z = I - I'$ we get our fundamental formulæ

$$\frac{dy}{d\xi} = (k + s)z\,; \quad \frac{dz}{d\xi} = k(y - 2E) \quad . \quad . \quad (2)$$

In the general case this k consists of the general absorption k_0 and the collision part of the line absorption $(\eta/1 + \eta)s_0$. (In Eddington's paper η denotes our s_0/k_0, his ϵ denotes our $k/s_0 = \eta/(1 + \eta)$; his dx is our $k_0 d\xi$; the other differences are the numerical coefficients produced by the oblique radiation, which would change the second members in our formulæ into $\frac{3}{2}(k + s)z$ and $2k(y - 2E)$.)

In the central parts of a Fraunhofer line the general absorption may be neglected compared with the line absorption. We will first treat this simpler problem. Now $k + s$ in (2) is equal to s_0, and we will put $s_0 d\xi = dx$ (special optical depth). Since the variable absorption coefficients depend on electron pressure we have to express the general optical depth τ that determines the temperature gradient, in this special co-ordinate x, by means of their relations to the electron pressure.

In $dx = s_0 d\xi = -s_0 \rho dh$, ρ denotes the total density, and the relative abundance of the atoms producing our Fraunhofer line is included in the line-absorption coefficient s_0. To introduce the electron pressure P we substitute $\rho dh = -dp/g$, and for the ratio of total

pressure to electron pressure, which we suppose to be constant through the atmosphere (*vide* 2 (*f*)), we write $p/\mathrm{P} = dp/d\mathrm{P} = p'$. Then we have $d\xi = p'd\mathrm{P}/g$. The general absorption coefficient k_0 is assumed to be proportional to P, thus $k_0 = \kappa_0\mathrm{P}$. The same holds then for $\bar{k}$, which chiefly consists of k_0. Putting $\bar{k} = \bar{\kappa}\mathrm{P}$, we have

$$d\tau = \bar{k}d\xi = \bar{\kappa}\frac{p'}{g}\,\mathrm{P}d\mathrm{P}\,; \quad \tau = \frac{\bar{\kappa}}{2}\frac{p'}{g}\mathrm{P}^2.$$

Introducing now the special optical depth x, we have

$$dx = s_0 d\xi = s_0\frac{p'}{g}d\mathrm{P}\,; \quad \mathrm{P} = \frac{g}{p's_0}x\,; \quad \tau = \frac{\bar{\kappa}}{s_0^2}\frac{g}{p'}\frac{x^2}{2}.$$

We treat κ_0 and $\bar{\kappa}$ as constants ; this means that in going downward in the atmosphere the change of the absorption coefficient with the increasing electron pressure is represented in our formulæ, but not its variation caused by the increase of temperature at greater depths.

The black-body emission E can now be expressed in terms of x,

$$2\mathrm{E} = 2\mathrm{E}(\lambda, 0)(1 + c\tau) = 2\mathrm{E}(\lambda, 0)\Big(1 + \frac{1}{2}c\frac{\bar{\kappa}}{s_0^2}\frac{g}{p'}x^2\Big).$$

Putting

$$\frac{\kappa_0}{s_0^2}\frac{g}{p'} = a \qquad . \qquad . \qquad . \qquad . \qquad (3)$$

this is transformed into

$$2\mathrm{E} = 2\mathrm{E}_0\Big(1 + \frac{1}{2}c\frac{\bar{\kappa}}{\kappa_0}ax^2\Big).$$

For the division of the line absorption into a monochromatic and a black-body part we introduce two quantities Q and q by

$$\frac{k}{s} = \frac{\mathrm{P}}{\mathrm{Q}} = \frac{x}{q}\,; \quad q = \frac{p'}{g}s_0\mathrm{Q} = s_0\frac{\mathrm{Q}}{\mathrm{P}}\int\rho dh \qquad . \qquad . \qquad (4)$$

Q is the electron pressure, q the corresponding special optical depth for which half the energy of the excited atom is removed by collisions, half is radiated, *i.e.* for which $\eta = 1$. Q may be considered as a constant for a certain line ; q is proportional to s_0 and varies rapidly over the absorption line, being very large in the centre and decreasing to 0 at the extreme wings ; neither of them changes with the depth of the layer. Introducing these quantities the equations (2) take the form

$$\frac{dy}{dx} = z\,; \quad \frac{dz}{dx} = \frac{x}{x+q}\Big\{y - 2\mathrm{E}_0\Big(1 + \frac{1}{2}c\frac{\bar{\kappa}}{\kappa_0}ax^2\Big)\Big\} \qquad . \qquad (5)$$

For the wings of a Fraunhofer line the only difference is that in k we have to add the general absorption $k_0 = \kappa_0\mathrm{P} = \frac{\kappa_0}{s_0}\frac{g}{p'}x$. Then the equations

$$\frac{dy}{d\xi} = (k_0 + s_0)z\,; \quad \frac{dz}{d\xi} = \Big(k_0 + s_0\frac{x}{x+q}\Big)(y - 2\mathrm{E})\,;$$

take the form

$$\frac{dy}{dx} = (1 + \alpha x)z\,; \quad \frac{dz}{dx} = \left(\frac{x}{x+q} + \alpha x\right)\left\{y - 2E_0\left(1 + \frac{1}{2}c\frac{\bar{\kappa}}{\kappa_0}\alpha x^2\right)\right\} \qquad (6)$$

An alternative form of these equations is obtained by putting $\alpha x = u$:

$$\frac{dy}{du} = \frac{1}{\alpha}(u+1)z\,; \quad \frac{dz}{du} = \frac{1}{\alpha}\left(u + \frac{u}{u+\alpha q}\right)\left\{y - 2E_0\left(1 + \frac{1}{2}c\frac{\bar{\kappa}}{\kappa_0}\frac{u^2}{\alpha}\right)\right\} \qquad (6^*)$$

The boundary conditions, for (5) as well as for (6), are that at the surface, for $x = 0$, $y = z$, and at great depths, for $x \to \infty$, $y - 2E$ disappears.

5. The equations (5) and (6) cannot be integrated in a finite form. For small values of x development in series is possible ; since, however, one of the boundary conditions stands for $x = \infty$, this cannot give a practical solution. After trying different means, the least laborious method was found to consist in simple mechanical integration, beginning with assumed values of $y_0 = z_0$ for $x = 0$, and varying them till the desired result for $x \to \infty$ was obtained. With appropriate small steps, Δx, in x the equations (5) take the form

$$y_{n+1} = y_n + z_{n+1/2}\Delta x\,; \quad z_{n+1/2} = z_{n-1/2} + \frac{x_n}{x_n + q}\Delta x(y_n - 2E)$$

and y_{n-1}, $z_{n-1/2}$, y_n, $z_{n+1/2}$, y_{n+1}, etc., are computed in succession. The variables y, z and their common zero value $y_0 = z_0$ consist of two terms, one expressed in $2E_0$, the other in $E_0 c\frac{\bar{\kappa}}{\kappa_0}\alpha$; the numerical coefficients are calculated separately. The first-named one approaches, with increasing x, to 1 for y, to 0 for z ; if the initial zero value y_0 is too large, then the numerical y reaches 1 and passes this value, while z returns and begins to increase before reaching 0 ; if y_0 is too small, then x reaches 0 and passes it, while y returns and decreases before reaching 1. At great depths the values belonging to adjacent y_0 are strongly diverging ; thus by repeated interpolation it is possible to find the right y_0 with sufficient precision. In the second term the numerical value of $y - x^2$ has also to decrease asymptotically to 0 ; if y_0 deviates from the true value, $y - x^2$ either returns and goes up increasingly before reaching 0, or passes through 0, decreasing with growing rapidity.

The numerical results of the solution of equations (5) are given in Table I. It is extended to low values of q, though here the general absorption comes into play and equations (6) should be used. The second term is expressed in $E_0 c\frac{\bar{\kappa}}{\kappa_0}\alpha$, where α, because it contains $1/{s_0}^2$, is proportional to the inverse square of q. The product $E_0 c\frac{\bar{\kappa}}{\kappa_0}\alpha q^2$ is independent of s_0 and constant over the breadth of the absorption line ; hence the second term is given in Table I. also as the numerical coefficient of this constant. Here it appears that for large values of q, *i.e.* for strong absorptions in the central parts of the line, the influence

of the second term disappears, because only radiation from the uppermost layers of constant temperature is emitted from the surface.

Table I.

q.	Coeff. of E_0.	Coeff. of $E_0 c\frac{\bar{\kappa}}{\kappa_0}\alpha$.	Coeff. of $E_0 c\frac{\bar{\kappa}}{\kappa_0}\alpha q^2$.
0	1·0	1·0	..
0·1	·877	1·141	114·1
0·5	·808	1·383	5·53
1	·744	1·67	1·67
5	·551	2·58	·103
10	·486	3·35	·0335
50	·325	5·98	·00239

6. For very large values of q direct computation becomes difficult. In this case x remains small and the second term is imperceptible. Because in this case the coefficient $x/(x+q)$ may be replaced by x/q a direct solution of equation (5) in the form of a definite integral is possible. The equation

$$\frac{d^2y}{dx^2} = \frac{x}{q}(y - 2E_0)$$

has the solution

$$y - 2E_0 = A\int_0^1 (e^{tp} + e^{-tp})(p^2 - 1)^{5/6}dp + B\int_1^\infty e^{-tp}(p^2 - 1)^{-5/6}dp,$$

where $t = \frac{2}{3}q^{-1/2}x^{3/2}$. The boundary condition $y - 2E_0 = 0$ for $t = \infty$ makes $A = 0$. Then we have for z,

$$z = \frac{dy}{dx} = -Bq^{-1/2}x^{1/2}\int_1^\infty e^{-tp}(p^2 - 1)^{-5/6}pdp,$$

which by the substitution $pt = u$ becomes

$$z = -B\left(\frac{3}{2q}\right)^{1/3}\int_t^\infty e^{-u}(u^2 - t^2)^{-5/6}udu.$$

For $x = 0$, $t = 0$ these integrals are

$$y_0 - 2E_0 = B\int_1^\infty (p^2 - 1)^{-5/6}dp = BI_1$$

$$z_0 = -B\left(\frac{3}{2q}\right)^{1/3}\int_0^\infty e^{-u}u^{-2/3}du = -Bq^{-1/3}I_2.$$

The other boundary condition $y_0 = z_0$ determines B and y_0,

$$B = -\frac{2E_0}{I_1 + q^{-1/3}I_2}; \quad y_0 = z_0 = 2E_0\frac{I_2}{I_2 + I_1\sqrt[3]{q}}.$$

From the two numerical integrals I_1 and I_2 appearing in these formulæ,

one is a gamma function, and the other can be calculated easily by numerical integration,

$$I_2 = \sqrt[3]{\tfrac{3}{2}}(\Gamma\tfrac{1}{3}) = 3{\cdot}066$$

$$I_1 = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt[3]{\cos\phi \sin^2\phi}} = 4{\cdot}106 = 1{\cdot}339 \times 3{\cdot}066.$$

Hence the limiting value of $y_0 = z_0$ for large values of q is given by

$$y_0 = z_0 = \frac{2E_0}{1 + 1{\cdot}339\sqrt[3]{q}} \quad . \quad . \quad . \quad (7)$$

Comparing this formula with some values computed by numerical integration, we find for

	Numerical.	Formula.	Difference.
$q=5$	·566	·608	−·042
10	·486	·515	−·029
20	·413	·432	−·019
50	·325	·337	−·012

Thus it appears that for $q > 50$ the formula can be used without great error.

It may be remarked that for very strong absorption coefficients the residual intensity varies as the inverse third root of the coefficient of absorption. With only a constant general black absorption Milne found the residual intensity varying as the inverse square root of the line absorption.

7. The solution of formulæ (6), which have to be used for the wings of the absorption line, is little more laborious than that of formulæ (5). It takes, however, more time because now the second variable parameter α is not restricted to a second independent term, and for each different α a new series of solutions has to be made. For one absorption line we may assume the product αq^2 to be constant :

$$q = \frac{p'}{g} s_0 Q\,; \quad \alpha = \frac{\kappa_0}{s_0^2}\frac{g}{p'}\,; \quad \alpha q^2 = \kappa_0 \frac{p'}{g} Q^2\,;$$

the rapidly changing s_0 has disappeared. The quantity αq^2 may be written $k_0(Q) \times \xi(Q)$, *i.e.* the coefficient of general absorption at electron pressure Q (for which $\eta = 1$), multiplied by the homogeneous depth ξ for which the electron pressure is Q. These factors depend on physical probability coefficients, different for different atoms and lines, and contain, moreover, parameters of the star. Hence different sets of solutions must be made for different values of αq^2.

In order to find the order of magnitude of this αq^2 we have made an estimate of k/s, the ratio of the number of effective collisions and the number of free transitions to the lower state for an excited atom. For the first-named number we put $\pi n_e v_e \sigma_a^2$ (number and velocity of the

electrons, effective radius of the atoms), for the other $1/\tau$ (mean lifetime). With the numerical values (for T = 6000°) $n_e = 1{\cdot}2 \times 10^{12}$P, $v_e = 4{\cdot}3 \times 10^7$, $\sigma = 10^{-8}$, $\tau = 10^{-8}$, we find $k/s = 1{\cdot}6 \times 10^{-4}$P; hence Q, the electron pressure, for which $k/s = 1$, is nearly 10^4 (*i.e.* 10^{-2} atmos.). The general absorption in the sun's photosphere was taken from Milne's computation *; from the value $0{\cdot}78 \times 10^4$, standing for P = 10^2, we derive $k_0(Q) = 0{\cdot}8 \times 10^6$. The other factor $\xi(Q) = p'Q/g = p(Q)g$, the total pressure at that Q level, divided by the gravity, 3×10^4; since this total pressure is somewhat larger than the electron pressure Q itself, this second factor is of the order 1. Hence for the order of magnitude of the quantity aq^2 we find 10^6 in the case of solar conditions. For giant stars with smaller g the second factor will be larger; for hotter stars the first factor, in consequence of the smaller absorption coefficient, may be smaller. It should be considered, moreover, that especially owing to the uncertainty of σ, the "effective" atomic radius, our estimate could only be a very rough one.

The results of the numerical integration of (6) are found in Table II.; for $aq^2 = 10^6$ they have been computed most completely; for $aq^2 = 1$ and $aq^2 = 10^8$ only for a smaller number of values, to see the variation of the curve. The coefficients of E and of $E\frac{\bar{\kappa}}{\kappa_0}c$ are given separately, in order that the resulting intensity may be computed for different values of c (wave-length and temperature) and different suppositions for $\bar{\kappa}/\kappa_0$. That the Fraunhofer lines in total take away 15 per cent. of the spectral energy does not mean that κ_0 amounts to 85 per cent. of the total absorption coefficient $\bar{\kappa}$; for the strongest monochromatic absorptions act on radiations already strongly absorbed. Hence the quotient $\bar{\kappa}/\kappa_0$ probably is considerably larger than 100/85, but its real amount is quite uncertain. Simply in order to obtain results, which can be used as examples for further deductions, I have computed the residual intensities r for $c\bar{\kappa}/\kappa_0 = 1{\cdot}5$ (*i.e.* for the sun, with $\bar{\kappa}/\kappa_0 = 1$, $\lambda = 4000$ A.; for $\bar{\kappa}/\kappa_0 = 1{\cdot}5$, $\lambda = 6000$ A.); they are expressed with the continuous background intensity $E_0(1 + c\bar{\kappa}/\kappa_0)$ taken = 1.

Table II.

q.	a. ($aq^2=1$)	Coeff. of E_0.	Coeff. of $E_0 c\bar{\kappa}/\kappa_0$.	r.
·05	400	0·985	0·953	0·966
·1	100	·971	·911	·931
·2	25	·947	·833	·878
·5	4	·883	·650	·744
1	1	·808	·451	·594
2	·25	·715	·256	·439
5	·04	·586	·085	·286
10	·01	·497	·031	·217

* "Absorption Coefficients and Pressure of Radiation," *M.N.*, **85**, 773, 1925.

TABLE II.—*continued.*

q.	a.	Coeff. of E_0.	Coeff. of $E_0 c\bar{\kappa}/\kappa_0$.	r.
	$aq^2=10^6$			
10	10^2	0·995	0·992	0·993
10^2	10^2	·958	·924	·938
2×10^2	25	·920	·858	·883
5×10^2	4	·829	·700	·752
10^3	1	·702	·527	·597
2×10^3	·25	·574	·340	·434
5×10^3	·04	·384	·150	·244
10^4	·01	·268	·0700	·149
2×10^4	25×10^{-4}	·180	·0304	·0903
5×10^4	4×10^{-4}	·103	$·943\ 10^{-2}$	·0470
10^5	10^{-4}	·0675	$·376\ 10^{-2}$	·0292
10^6	10^{-6}	·0182	$·150\ 10^{-3}$	·0074
10^7	10^{-8}	·00697	$·388\ 10^{-5}$	·0028
10^8	10^{-10}	·00315	$·87\ 10^{-7}$	·0013
	$aq^2=10^8$			
10^3	10^2	0·958	0·924	0·938
10^4	1	·714	·528	·602
10^5	10^{-2}	·268	·0704	·149
10^6	10^{-4}	·0657	$·391\ 10^{-2}$	·0287
10^7	10^{-6}	·0152	$·180\ 10^{-3}$	·0062
10^8	10^{-8}	·00395	$·691\ 10^{-5}$	·0016
10^9	10^{-10}	·00148	..	·00059
10^{10}	10^{-12}	·000686	..	·00027

The numerical results show that *for moderate absorptions the residual intensity depends only on* a, and the differences of q have hardly any influence. Even for the widely different supposition of $aq^2 = 1$ the two separate coefficients deviate, but the resulting intensity does not. This means that the collision part of the monochromatic coefficient in these cases is negligible compared with the general absorption. At the bottom of the tables, however, for very low residual intensities, the matter is different; here the values for the same a decrease with increasing aq^2 and they approach to a dependence on q chiefly. The low residual intensities themselves depend only on the uppermost atmospheric layer; the value of r, however, depends on $c\bar{\kappa}/\kappa_0$ because it is the ratio of that low intensity to the continuous background $E_0(1 + c\bar{\kappa}/\kappa_0)$ produced by the deep layers. For very large q the formula (7) of § 6, expressing the condition that the general absorption

may be neglected compared with the collision part of the monochromatic absorption, gives the values

$q = 10^6$	0·0148
10^7	·00692
10^8	·00321
10^9	·00149
10^{10}	·00069,

asymptotically coinciding with the values of the above tables.

8. To derive by means of these tables the contours of an absorption line it is now necessary to know the variation of a and q, *i.e.* of the complete monochromatic absorption coefficient s_0 in the vicinity of the wave-length λ_0. A formula for this coefficient has been derived by W. Voigt.* His formula (34), by substituting $s_0\rho = \frac{4\pi}{\lambda_0} n\kappa$ and $a\sqrt{1+y^2} = \alpha$, takes the form

$$s_0\rho = \frac{4\sqrt{\pi}\epsilon^2 N}{mcb}\int_0^\infty y e^{-y^2} dy \text{ arc tan } \frac{2aby}{\mu^2 + a^2 - b^2y^2},$$

where ϵ is the charge, m the mass of an electron, c the velocity of light, N the number of atoms producing the line per c.c., $b = 2\pi\nu_0 w/c^2$, $w^2 = 2k\mathrm{T}/m_a$ (m_a is the mass of the atom, k the Boltzmann constant, w the mean Maxwellian velocity), $\mu = 2\pi(\nu - \nu_0)$. The quantity α is composed of two parts, $\alpha = \frac{1}{2}\nu' + 1/\tau$, the first representing the electromagnetic damping, the second the influence of collisions, τ being the mean free running time. For the first term the formula

$$\nu' = \frac{8\pi^2\epsilon^2\nu_0^2}{3mc^3},$$

used by R. Minkowski,† gives the value $\log \nu' = 8{\cdot}143$. For the second term the formula $1/\tau = \pi n\sigma^2 v\sqrt{2}$, taking $\mathrm{T} = 6000°$, $n = 10^{12}p = 10^{14}$, v(electr.) $= 4 \times 10^7$, $\sigma = 10^{-8}$, gives 2×10^6. So under the conditions existing in stellar atmospheres the collision term may be neglected compared with the other, and in Voigt's integral $\alpha = \frac{1}{2}\nu'$ is a constant. Computation shows that for hydrogen at 6000° $\log b = 11{,}193$, and for a difference $\nu - \nu_0$ corresponding to 1 A. (at λ4000) $\log \mu = 12{,}071$; hence the proper width of the line ν' is much smaller than the Doppler width b, while the latter is smaller than one angstrom. In evaluating the integral we neglect a^2 in the denominator, and put

$$y = \frac{\mu}{b}\tan\psi; \qquad \frac{\nu'}{2\mu}\tan 2\psi = \tan\phi;$$

* W. Voigt, "Ueber das Gesetz der Intensitätsverteilung innerhalb der Linien eines Gasspektrums," *Sitz. Ber. München*, 1912, p. 603.

† *Zeitschr. für Physik*, **36**, 839 (1926).

then the integral takes the form

$$\int_0^\infty \phi y e^{-y^2} dy.$$

When y and $\tan\psi$ are increasing regularly from 0 to ∞, 2ψ increases from 0 through $\frac{1}{2}\pi$ (for $y = \mu/b$) to π; because $\nu'/2\mu$ is very small (except for very small μ) ϕ increases from 0 at first slowly, then from a small value jumps rapidly through $\frac{1}{2}\pi$ to nearly π, and remains so during the remaining region of integration. We therefore divide the integral into two parts by the limit $y = \mu/b$; in the first part we may put $\phi = \nu'\psi/\mu$, in the second part $\phi = \pi$. The omitted contributions to the first integral by the part next to the upper limit $y = \mu/b$ are of the order of magnitude $\frac{\nu'}{\mu} e^{-(\mu/b)^2}$; moreover, they cancel out for the main part by the negative contributions $\phi - \pi$ near the lower limit of the second integral. Thus the integral becomes

$$\int_0^{\mu/b} \frac{b\nu'}{\mu^2} y^2 e^{-y^2} dy + \pi \int_{\mu/b}^\infty y e^{-y^2} dy = -1/2 \frac{\nu'}{\mu} e^{-\mu^2/b^2} + \frac{1}{2}\frac{b\nu'}{\mu^2} \int_0^{\mu/b} e^{-y^2} dy + \frac{\pi}{2} e^{-(\mu/b)^2}.$$

In the case of large μ/b the integral in the second term is nearly $\frac{1}{2}\sqrt{\pi}$; in this case the first term is extremely small and the whole expression may be written

$$1/4\sqrt{\pi}\frac{b\nu'}{\mu^2} + \tfrac{1}{2}\pi e^{-(\mu/b)^2}.$$

For decreasing μ the first term has to be multiplied by a factor which slowly decreases from unity and for $\mu = b$ reaches 0·84. For very small μ ($\mu < b$) the first of the two integrals may be developed

$$\frac{\nu'}{b^2}\mu\left(\frac{1}{3} - \frac{1}{5}\frac{\mu^2}{b^2} + \ldots\right).$$

For the centre of the line ($\mu = 0$) Voigt's integral takes the form

$$\int_0^\infty 2ye^{-y^2} dy \operatorname{arc\,tan} \frac{2by}{\nu'},$$

which, by the same process of dividing, gives the result

$$1/6\left(\frac{\nu'}{b}\right)^2 + 1/2\pi e^{-(\nu'/2b)^2}.$$

Since ν'/b in all practical cases is small, this reduces to $\frac{1}{2}\pi$, the same result as is obtained by putting $\mu = 0$ in the general solution.

The result for the absorption coefficient is now

$$\left.\begin{aligned} s_0\rho &= \frac{2\sqrt{\pi}\epsilon^2 N}{mcb}\left\{\frac{\sqrt{\pi}}{2}\frac{\nu' b}{\mu^2} + \pi e^{-(\mu/b)^2}\right\} \qquad (\mu > b) \\ \text{or} \qquad &= \frac{2\sqrt{\pi}\epsilon^2 N}{mcb}\left\{\frac{2}{3}\frac{\nu'}{b^2}\mu + \pi e^{-(\mu/b)^2}\right\} \qquad (\mu < b) \end{aligned}\right\} \quad . \quad (8)$$

It consists of two separate parts. One part is the Doppler broadened line, from the centre decreasing as a Gaussian curve with a modulus of width b, corresponding to the mean velocity of the atoms, equal to $\cdot 13\sqrt{\frac{1}{\mu_a}\frac{T}{6000}}$ angstrom (μ_a denotes the atomic weight). The factor b in the denominator shows that the maximum value of this term increases with decreasing width. The second part, the resonance wing of the line, proportional to $1/\Delta\lambda^2$, is identical with Unsöld's formula (1); it depends only on general physical constants and is the same for all kinds of atoms:

$$s_0\rho = \frac{2\pi\epsilon^4 N}{3m^2c^4}\frac{\lambda_0^2}{\Delta\lambda^2} \quad . \quad . \quad . \quad . \quad (9)$$

Introducing numerical values (valid for hydrogen and calcium atoms at 6000° and λ 4000) we have (numbers in brackets denote logarithms, $\Delta\lambda$ expressed in angstroms)

$$\left.\begin{aligned}\frac{s_0\rho}{N} &= [0\cdot424 - 18]1/\Delta\lambda^2 \ (\mu > b)\\ \text{or}\quad &[0\cdot935 - 16]\Delta\lambda \quad (\mu < b)\end{aligned}\right\} + [0\cdot780 - 13]e^{-57\Delta\lambda^2} \qquad \text{(for } H\text{)},$$

$$\left.\begin{aligned}\frac{s_0\rho}{N} &= [0\cdot424 - 18]1/\Delta\lambda^2 \ (\mu > b)\\ \text{or}\quad &[0\cdot338 - 13]\Delta\lambda \quad (\mu < b)\end{aligned}\right\} + [0\cdot581 - 12]e^{-2280\Delta\lambda^2} \qquad \text{(for } Ca\text{)}.$$

A graphical representation of $\log s_0\rho/N$ (*vide* fig. 1, where each vertical division represents a factor 10) shows that in the outer parts the resonance term dominates and the Doppler term is imperceptible, while in the inner parts the latter dominates and the other is negligible, so that in each region simply one term has to be used, except for a narrow strip of transition, where the terms combine. This transition takes place at ·28 A. for hydrogen, at ·07 A. for calcium. It is clear from the above values that only an extremely small fraction of the emission of an atom is dispersed into the wings of a line.

9. We first consider the wings of strong lines, *i.e.* the cases where a moderate residual intensity is produced by the resonance part of the line-absorption curve. This residual intensity was found to depend solely on the quantity α; for $r = 0\cdot5$, according to Table II., we have $\log\alpha = -\cdot35$ ($\alpha = \cdot45$), hardly different for different assumptions regarding the constants of the atmospheric gases. Now $\alpha = (\kappa_0/s_0^2)(g/p')$ depends on the ratio of the general continuous absorption κ_0 and the monochromatic line absorption s_0, combined with some gravity and pressure relations of the atmosphere, which were introduced because κ_0 depends on the electron pressure and s_0 does not.

The coefficient of general absorption has been the subject of the well-known investigations by Eddington * and Milne,† making use of Kramers's theoretical work on X-ray absorption. The aim of these

* *M.N.*, **84**, 104 (1924).
† *Ibid.*, **85**, 750, 768 (1925).

researches was the derivation of the mean absorption coefficient for the total radiation in the stellar interior; the same formulæ were then applied by Milne to the atmospheric layers. This general coefficient forms the continuous part of the total coefficient k, introduced in our formulæ, § 2, which determines the temperature gradient. For the line contours we want κ_0, the continuous absorption coefficient for just the wave-length of the line considered. In his paper on

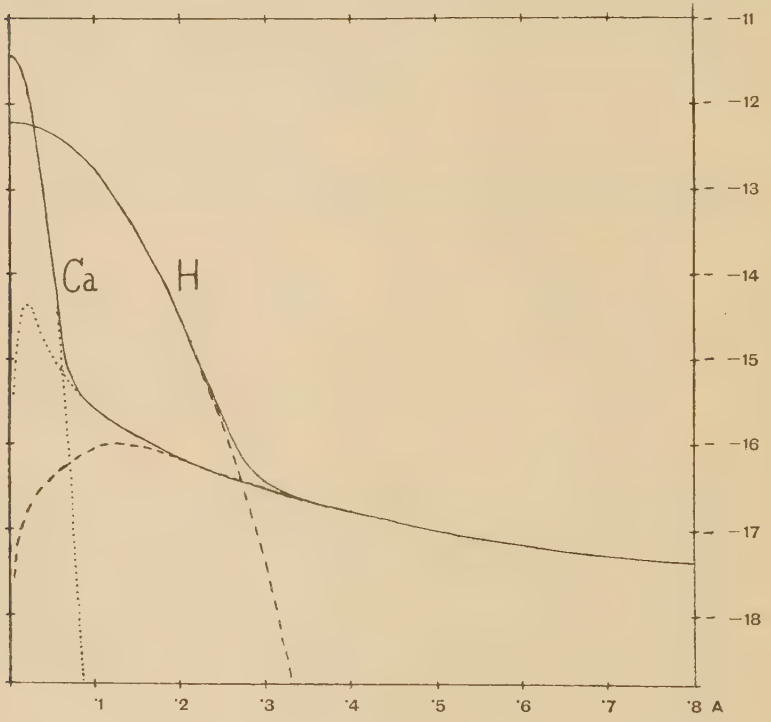

Fig. 1.

"Continuous Absorption," * J. A. Gaunt has given a new derivation of the basic formulæ by wave mechanics. In the application of these formulæ to the case of a stellar atmosphere there are a number of uncertain factors; we discuss them in the Appendix (§ 13) to the present paper. The result may be put into the form

$$\kappa_0 = \frac{4\sqrt{6\pi}}{9} \frac{\epsilon^6 a'}{hc(mk\mathrm{T})^{3/2}\mu_a m_{\mathrm{H}}} \frac{f(\nu\nu_1)}{\imath\hat{}} e^{h\nu/k\mathrm{T}} \quad . \quad . \quad (10)$$

where a' is the relative concentration in mass (fraction of the total mass) of the active atoms, μ_a is their atomic weight, and $f(\nu\nu_1)$ denotes

* *Phil. Trans.*, A, **229**, 163 (1930).

the unimportant factor $\left(\frac{\nu}{\nu_1} - \frac{kT}{h\nu_1}\right)$ with a numerical value of nearly 0·4. Taking, according to that discussion, $a' = 1/200$, $\mu_a = 40$, we find for $T = 6000°$ and wave-length 4000 A. the numerical value $\log \kappa_0 = -3{\cdot}8$.

For the resonance part of the line-absorption coefficient we found

$$s_0\rho = \frac{2\pi\epsilon^4}{3m^2c^4} N \left(\frac{\lambda_0}{\Delta\lambda}\right)^2.$$

Introducing n, the effective concentration of the atoms producing this line (fraction of the total mass), we may write $N = n\rho/\mu m_H$, and

$$s_0 = \frac{2\pi\epsilon^4}{3m^2c^4 m_H} \frac{n}{\mu}\left(\frac{\lambda_0}{\Delta\lambda}\right)^2 = [-1{\cdot}00]\frac{n}{\mu}\left(\frac{\lambda_0}{\Delta\lambda}\right)^2.$$

Hence

$$a = \frac{\sqrt{6\pi}}{\pi^2} \frac{m^{5/2} m_H c^4}{\epsilon^2 h^{3/2}} \frac{a'}{\mu_a} \frac{g}{p' T^{3/2}} \lambda_0^3 f(\nu\nu_1) e^{c_2/\lambda_0 T} \left(\frac{\mu}{n}\right)^2 \left(\frac{\Delta\lambda}{\lambda_0}\right)^4 . \quad (11)$$

In this expression the first two groups are numerical or physical constants ; the quantities of the third group, depending on the constitution of the atmosphere, are rather uncertain and have to be assumed in some way ; the next group consists of star parameters, whereas μ, n, and λ_0 are different for different absorption lines. It is the fractional concentration or abundance n which by means of this formula is determined from the observed $\Delta\lambda$.

In Unsöld's treatment of the problem the line width depends on the absolute number of absorbing atoms above the photosphere, and is used to derive this number, because the simplifying assumption is made of a continuously emitting black surface at the bottom with an overlying absorbing atmosphere. In reality the continuously emitting matter is composed of the very atoms that by their absorption produce the Fraunhofer lines, and all the atmospheric layers contribute to both the continuous emission and the line absorption. It is obvious that in such a case the strength of an absorption line does not depend on the number of active atoms themselves, but on their fractional concentration n relative to the total mass. The case is different for the centre of the line, where the residual intensity is nearly zero and depends on q, *i.e.* on s_0. In this case the absorption takes place in the uppermost layers of the atmosphere ; hence the case of a deeper continuously emitting source and another overlying absorbing atmosphere is realised here.

The half-width of a line is found by computing $\Delta\lambda$ for $a = 0{\cdot}45$. Then our formula (11) shows the following relations :—

(1) For the same element $\Delta\lambda \sim \sqrt{n}$, where n is proportional to the weight of the transition producing the line. This corresponds to Unsöld's result that $\Delta\lambda \sim \sqrt{N}$.

(2) For different elements in the same atmosphere $\Delta\lambda \sim \sqrt{n/\mu}$, where n/μ denotes the relative number of active atoms.

(3) For giant stars with low g the lines are wider than for dwarf stars in the inverse ratio $\sqrt[4]{g}$.

(4) For hotter stars the lines are wider than for cooler stars in the ratio $T^{3/8}$; the change of p' in the opposite direction does not compensate for this. This corresponds to Milne's result,* that by small gravity as well as by high temperature the width of an absorption line is increased.

We will now make use of the numerical values already computed for the case of the sun (T = 6000°) and wave-length 4000 A.:

$$\kappa_0 = [-3{\cdot}8];\ s_0 = [-1{\cdot}00]\frac{n}{\mu}\left(\frac{\lambda_0}{\Delta\lambda}\right)^2 = [6{\cdot}20]\frac{n}{\mu}\left(\frac{1}{\Delta\lambda}\right)^2 (\Delta\lambda \text{ in angstroms}).$$

For the sun $g = [4{\cdot}44]$; for $p' = p/P$, which in the case of strong ionization approaches to 1, we take 2, assuming an equal number of atoms and electrons; hence $g/p' = [4{\cdot}14]$. Then

$$a = [-12{\cdot}06]\left(\frac{\mu}{n}\right)^2 \Delta\lambda^4.$$

If for $\Delta\lambda$ we take the half-width of a line, where the residual intensity is 0·5, then $a = 0{\cdot}45$ (log $= -0{\cdot}35$), and we have

$$\frac{n}{\mu} = [-5{\cdot}86]\Delta\lambda^2.$$

For the Ca^+ lines H and K the half-widths are 4·5 A. and 6·5 A., or $\Delta\lambda^2$ is 20 and 40; for all the Ca^+ atoms we put $\Delta\lambda^2 = 60$, and we find $n/\mu = [-4{\cdot}08]$; $n = [-2{\cdot}48]$. The Ca^+ atoms, therefore, constitute 1/300 in mass of the atmospheric gases of the sun.

For other lines with other wave-lengths the numerical constants are somewhat different. In κ_0 the different factors containing ν partly compensate one another, so that, going from 4000 A. to 5890 A. and 6560 A. the value of log κ_0 decreases by 0·50 and 0·58; the logarithms of the wave-length themselves increase by 0·168 and 0·215, so that the numerical coefficient in the formula for n/μ becomes smaller in the ratio $[0{\cdot}25 + 0{\cdot}34] = [0{\cdot}59]$ and $[0{\cdot}29 + 0{\cdot}43] = [0{\cdot}72]$. For the sodium lines D_1D_2 we read the half-widths from a diagram given by Unsöld, viz., 0·18 A. and 0·29 A., and for Hα in the same way, 0·52 A.†; from a table of Minnaert for the last-named the probably better value 0·64 A. is found.‡ Then for sodium and hydrogen we have

$$\frac{n}{\mu} = [-6{\cdot}45](0{\cdot}032 + 0{\cdot}084) = [-7{\cdot}39] \quad n = [-6{\cdot}03] \quad (Na).$$

$$\frac{n}{\mu} = [-6{\cdot}58](0{\cdot}41) \qquad = [-6{\cdot}97] \quad n = [-6{\cdot}97] \quad (H).$$

For sodium this n denotes the concentration of neutral atoms; from the ionization formula the non-ionized fraction for T = 6000° and

* *M.N.*, **89**, 17, 157 (1928).
† *Zeitschr. für Physik*, **46**, 772 and 778.
‡ *Ibid.*, **45**, 613.

P = 10^{-4} atm. is [− 2·6], so that the sodium atoms in total constitute 1/2500 of the solar atmosphere. In this case, however, our treatment is not adequate, because the non-ionized fraction decreases regularly with decreasing electron pressure; hence s_0 in our formulæ should have been taken varying with P. For hydrogen *Hα* comprises nearly 0·8 of all the transitions proceeding from the second quantum state; the concentration of hydrogen atoms in this state accordingly is given by [− 6·87]. In the case of thermodynamic equilibrium the fraction of hydrogen atoms in the second state for 6000° is given by [− 8·5]; since the total abundance of hydrogen atoms cannot exceed 1, our result confirms the well-known phenomenon that the Boltzmann equilibrium formula gives too low numbers for the higher quantum states.

Unsöld has deduced the half-widths of several Fraunhofer lines from photometric measurements made at the Einsteinturm in Potsdam.* He did not publish these values themselves, but only the number of atoms *NH* deduced from them. They are proportional to our n, but for the wave-length factor in κ_0. Comparing with our Ca^+ value and taking account of these factors we find the following mass-fractions :—

	Na_0.	Al_0.	$Ca+$.	$Sr+$.	$Ba+$.	$Ca_0(4226)$.	$Sr_0(4607)$.
$10^6 n$	·94	9·8	3300	2·6	·32	3·9	·009

All values deduced here depend on the value assumed for the continuous absorption coefficient and share its uncertainty. In § 13 reasons are given to expect that the coefficient is larger to an unknown amount, with an upper limit of 50 times. In this limiting case κ_0 = [− 2·1], and all the concentrations computed are increased 7 times, *i.e.* for Ca^+ we have n = 1/43 (log − 1·63), for *Na* 1/380, for *H* (2nd state) [− 6·02].

10. We have now to derive the theoretical contours of an absorption line. As an example we take a line produced by Ca^+ by a percentage of ·0016 (intermediate between the percentages producing the K and the H line). For each $\Delta\lambda$ we find $s_0\rho/N = s_0\mu m_H/n$ from formula (8), § 8. Then we compute $a = \kappa_0 g/p' s_0^2$. To derive the residual intensities r we make use of the tables of § 7, where the results for $c\bar{\kappa}/\kappa_0$ = 1·5 are used to construct a diagram, by means of which $r = f(\log a)$ can be read.

The resulting contour curve is represented in fig. 2 † by the dotted line, which farther from the centre coincides with the dashed and the full line. It has a half-width of 5·3 A., between the values for *H* and *K*. It is funnel-shaped and has an intensity zero for $\Delta\lambda$ = 0. It is entirely determined by the resonance part of the line-absorption coefficient. If now the percentage is diminished the line becomes narrower and its contour keeps the same figure if only the scale is widened proportionately to $\sqrt{n}$ (and so the total quantity of energy lacking and the effective

* *Zeitschr. für Physik*, **46**, 772 (1928).

† If the continuous absorption coefficient should be increased 50 times, all the logarithmic concentrations indicated in this figure should be increased by ·8; hence they would be −10 to −2 in fig. 2, −11·6 to −3·6 in fig. 3.

line width decrease with $\sqrt{n}$ also). But now the centre between $\pm$ 0·1 A. becomes gradually distorted by the Doppler part of the line-absorption coefficient. In the figure the curve for $n = 10^{-4.8}$ is represented by the dashed line on a 10-times wider scale; hence this curve for the main part is identical with the first one, and only in the

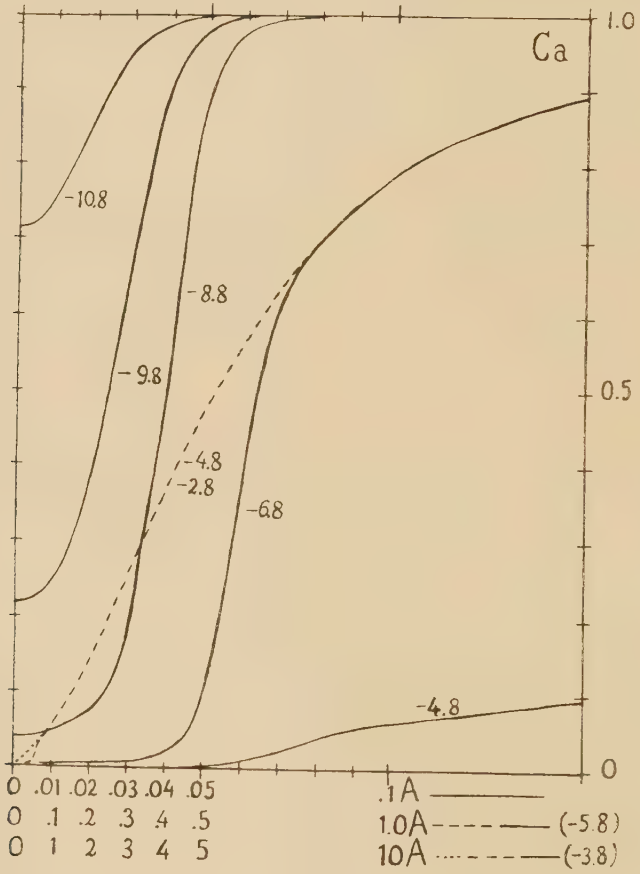

FIG. 2.

central part it is broadened and sack-shaped. For $n = 10^{-6.8}$ the intensity curve represented on a 100-times wider scale by a full line shows a much greater change; down to 0·08 A. with $r > 0{\cdot}7$ the curves coincide in shape; then the narrow curve takes the form of a cup, at $\Delta\lambda = {\cdot}05$ A. rapidly decreasing to nearly zero and remaining so. If now the percentage of active atoms is decreased still more (each curve is designated by the negative log of n), the absorption line is entirely determined by the Doppler curve; it contracts somewhat in

width, till at last it appears only ·05 A. wide, and its central intensity increases, being ·04 for $n = 10^{-8.8}$, ·21 for $n = 10^{-9.8}$, ·71 for $n = 10^{-10.8}$.

For other elements the contours have the same character and only the dimensions are somewhat different. With increasing atomic weight the Doppler curves contract in width ; for higher temperatures they become wider. For the extreme case of hydrogen the contours

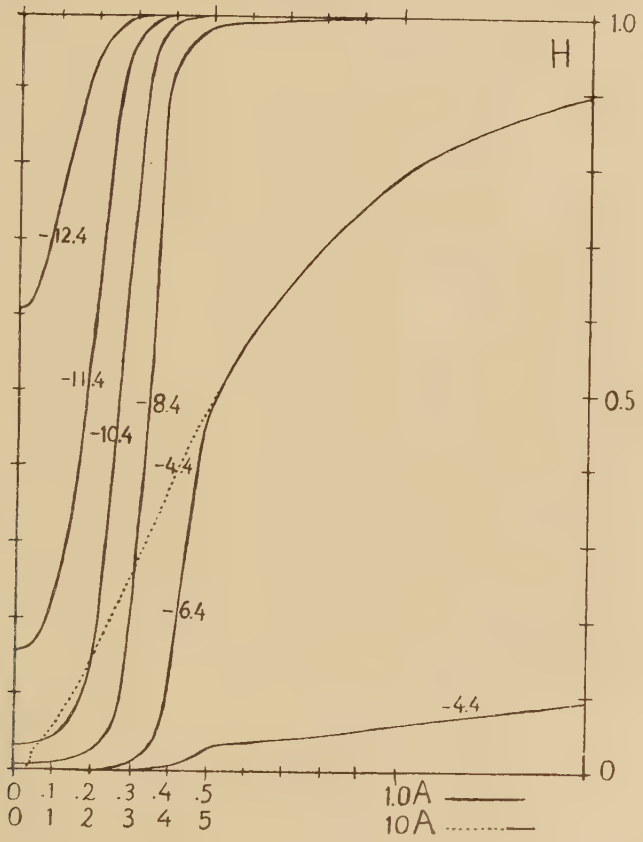

Fig. 3.

are represented in fig. 3 ; for identical wing width n is here 1/40 of the calcium value. Here the cup-shaped central Doppler part appears at half-width $\Delta\lambda = \cdot 5$ A., contracts to 0·3, while the central intensity remains a few per cent. only, and then, while growing dimmer by increasing central intensity, keeps a width of ·3 or ·4 A.

These results are at variance with what is obtained by approximate methods. Unsöld in his first researches found the central intensity of an absorption line zero, because he only treated the wings by means of the resonance formula for the line-absorption coefficient which for

$\Delta\lambda = 0$ gives a coefficient ∞. The central intensities observed are attributed by him to collisions of the radiating atoms, and he computes formulæ * for the residual intensity, depending on a quantity $\lambda = 2k/s$ in our notation. His conclusion that the central intensity for lines belonging to a multiplet must be the same, because k/s is the same for these lines, is not confirmed by our more exact computations. The divergence is not due to our general absorption k_0, for our formulæ without α (§ 5, Table 1, and § 6) give the same result: the residual intensity is there a function of $q = \frac{p'}{g} s_0 Q$, which contains not only Q (which expresses the ratio of collision and radiation for these lines), but also s_0, the total monochromatic absorption coefficient, which varies within a multiplet with the transition probability. It is easy to see that Unsöld's result follows from his assumption of one absorbing layer with definite k and s; in the real case, however, of a gradual decrease of density with height, the strongest lines take their origin in the highest and thinnest layers where collisions are less important than in the deeper layers where the faint lines originate. Moreover, the validity of many of his conclusions is impaired if we are right in our inference that for moderate residual intensities the general absorption far exceeds the collision part of the line absorption.

The same holds for Woolley's computation of a mean pressure from the central intensity. The central intensity produced by the atmosphere does not agree with that produced by an average layer of constant density. Woolley tried to estimate the blackening influence of the higher layers, but he takes only account of the chromosphere and so finds their influence insignificant.

11. There is a marked difference between the observed contours of absorption lines and the theoretical results derived here on the basis of the assumptions of § 2. Whereas for strong lines we find the centre of the line practically dark, such a dark centre is an exception among the observed lines. The only case mentioned is the supergiant ϵ Aurigæ, where Elvey states that during the eclipse the intensity in Hβ was zero in the centre (with a narrow emission line superposed).† On the drawing (p. 144 of his paper) $r = \cdot 15$ for $\Delta\lambda = \cdot 45$ A. is the faintest point measured, and it is possible that the more central part has the cup-shape indicated for H in our diagram, fig. 3. Everywhere else we find central intensities from ·07 (K line of Ca^+ in the sun),‡ up to some tenths, and even $> 0{\cdot}5$ for broad shallow lines in several A and B stars. The often-quoted influence of collisions of atoms and electrons in the solar atmosphere is, as we have seen now, unable to explain these high central intensities, so we have to look for other influences that may change the contours of the lines. We may separate them into four kinds or groups :—

(1) Changes in the basic physical quantities determining the absorption.

* *Sommerfeld Festschrift*, p. 95 (1928).
† *Astroph. J.*, **71**, 140 (1930).
‡ Minnaert, *Zeitschr. für Physik*, **45**, 610.

(2) Broadening of the absorption-coefficient curve.
(3) Broadening of the theoretical absorption-line contour by astronomical causes (size of the star, rotation).
(4) Change of the resulting contour by instrumental imperfections.

These will now be considered in turn.

12. (1) The first group is named here only to indicate that the proper width of the line ν', as we saw in § 8, will be increased for higher pressures by the influence of collisions. In such a case the resonance term in the formula for $s_0\rho$ increases, the same residual intensity in the wing corresponds to a lower percentage of active atoms, so that the maximum of the Doppler term is lowered, and the residual intensity in the centre is increased. A perceptible change for stellar absorption lines is not found on this account.

Holtsmark * investigated the mutual resonance influence of atoms of the same kind and found an appreciable increase of the proper width ν'; its amount is given by $1/4 \times 2{\cdot}63 \times 10^7 \sqrt{\mathrm{N}} r_0^{-3/2} \lambda/c$ (N, number of atoms per c.c.; r_0, atomic diameter), which for $r_0 = 10^{-7}$, partial pressure $= 10^{-6}$ atm., comes out $1{\cdot}4 \times 10^{10}$, much larger than ν'_2. Ornstein and Minnaert, however, found that this effect did not act as an increase of the proper width but as a broadening of the same kind as the Doppler effect. The quantities ν' and $1/\tau$ (§ 8) express a dissipation of energy, the basis of the absorption itself; since $\int s_0 d\lambda$ over the width of a line does not increase by this Holtsmark effect, it belongs to the next group of influences.

(2) Among the causes that broaden the curve of the line absorption coefficient s_0, we find in the first place the Stark effect, which has been shown by O. Struve † and A. Unsold ‡ to be present in the A-type and B-type stars and in the sun. Because the atoms are vibrating in the rapidly changing electric fields of the surrounding ions and electrons the lines split up into a number of components, which by the continual variation of the fields are diluted into broad bands. Holtsmark § has computed the distribution of intensity over one of these bands on the supposition that the dimensions of the atoms are negligible compared with their distances apart. R. Gans ‖ afterwards repeated the computations for the case of appreciable atomic dimensions; for the densities occurring in stellar atmospheres he found that no perceptible corrections were necessary. For the case of surrounding point sources (ions and electrons) Holtsmark finds the distribution of intensity as a function of the quantity $\Delta\nu/(2{\cdot}61 n^{2/3}\epsilon \mathrm{C})$ (n = number of point charges per c.c., ϵ their charge, C the constant of the Stark effect), and he gives a diagram in which it is represented by a curve. This curve in its main part resembles a Gaussian error curve, though in the extreme parts it does not fall off so rapidly as this curve; for practical computations we may use the expression $e^{-\Delta\nu^2/\alpha^2}$, where $\alpha = 1{\cdot}96 n^{2/3}\epsilon \mathrm{C}$. If such a

* *Zeitschr. für Physik*, **34**, 722 (1925).
† *Astroph. J.*, **69**, 173; **70**, 85, 237 (1929).
‡ *Zeitschr. für Physik*, **46**, 779; **59**, 353.
§ *Ann. d. Phys.*, **58**, 577 (1919). ‖ *Ibid.*, **66**, 396 (1921).

law of broadening is applied to a Doppler curve $e^{-(\mu/b)^2} = e^{-(2\pi\Delta\nu/b)^2}$ the result is a new Gaussian curve with exponent $\Delta\nu^2/(\alpha^2 + b^2/4\pi^2)$, while the maximum central amplitude is diminished in the ratio $b/2\pi\sqrt{\alpha^2 + b^2/4\pi^2}$. In our case, however, the matter is different. The light of central wave-length is coming from the highest atmospheric layers, where, owing to the small density, the Stark effect is absent; hence the top of the absorption-coefficient curve of fig. 1 is not widened at all; and the same holds for the whole Doppler part of the curve, for which the residual intensity is only a few hundredths. Only in the wings, where the moderate residual intensity is coming from deeper and denser layers, is the pressure broadening effective. The exact derivation of the line contour in the case of an appreciable Stark effect will be a complicated problem, because the different atmospheric layers, each with its own pressure effect, have to be combined. But so much we may say from our discussion, that in the centre of the line the residual intensity is *not* increased and only the wings are widened by this effect—at least, if the Stark effect in the wings is not so strong that they are diluted across the centre of the line. The total strength (effective width) of the line will then be increased. In the case of a very faint absorption line entirely produced by the Doppler part of the absorption, where the residual intensity in the centre is large already, the widening by the Stark effect will be accompanied by an increase of the central intensity, so that the contour is made shallower.

The Holtsmark effect, just mentioned, is also a pressure effect; its amount, as estimated above, comes out ten times smaller than b, so that in stellar atmospheres its influence probably is imperceptible.

Another cause of broadening of the absorption-coefficient curve would be given by irregular turbulent motions which were suggested by Unsöld * for the chromosphere. If the distribution of the velocities of these motions may be represented by an ordinary probability curve we will have a Doppler effect just as by the molecular translations but of a larger amount. Here we have a cause of broadening that affects the whole curve. If we represent it by $e^{-(2\pi\Delta\nu/\beta)^2}$, where $\beta = \nu_0 V/c$ (V = mean velocity), then the constant b in formula (8) for the line-absorption coefficient should be replaced by $\sqrt{b^2 + \beta^2}$, and the central top is lowered in the ratio $b/\sqrt{b^2 + \beta^2}$. Hence the central intensity in the absorption line is increased. The amount, however, is entirely insufficient; if this ratio, for example, is 1/10, so that the Doppler curve is ten times widened, the central intensity is displaced one number over our curves (figs. 2 and 3), increasing in the first numbers not at all, and then from 0·000 to 0·003, from 0·003 to 0·041, and from 0·041 to 0·214. Generally speaking, we cannot try to bring up the residual intensity in the centre to a moderate fraction by causes which broaden the absorption-coefficient curve, because then an impossibly large broadening would be required.

(3) The light emitted radially by a surface element of a regular

* *Astroph. J.*, **69**, 214 (1929).

stellar atmosphere, under the conditions stated in § 2, must exhibit absorption lines which are black in the centre, even if pressure and turbulence effects are taken into account. In the case of the Sun it is this radiation that can be investigated with the spectroscope. In the case of the stars we observe the total light of the star; central and border regions are superposed, and if there are differences in their spectrum the resulting image may become diffused and widened. The chief cause of widening is a rotation of the star. Its effect upon the contour of an absorption line was investigated by Carroll,* and by O. Struve and G. Shajn.† The radial velocity displacement is a linear function of the co-ordinate perpendicular to the projection of the rotation axis on the star disc; hence in the case of equal surface intensity the resulting contour of a line (of width o) is given by a semi-circle, and in the case of a darkening towards the limb it can easily be found by integrating the intensity along a strip parallel to the axis. This cause of broadening, because it works upon the absorption line itself, is more effective in raising the central intensity than the preceding group of causes which worked upon the absorption coefficient curve. Elvey ‡ could explain in this way the wide diffuse lines in a number of A-type and B-type stars, where the residual intensity in the centre reaches 0·60 to 0·90. In computing the rotational velocity, however, he assumes that in stars without rotation the central intensity of the line is as high as 0·30 to 0·40. These intensities have to be explained in another way.

(4) In stellar spectra the last-named group of causes, the lack of sufficient resolution and definition, may change the apparent contours of a line in a considerable degree. When a three-prism spectrograph is used, producing a scale of 10 A. to 1 mm. in the middle parts, the projected slit width usually corresponds to ·2 or ·3 A., which by diffraction is still further increased. Such narrow black central depressions as are represented in fig. 2 for *Ca* (which may be taken as a specimen for all the heavier elements), with a width of ·10 to ·05 A., cannot be distinguished in these spectra. They are diffused and the black centre is effaced. Computing the apparent central intensity in the case in which the light is diffused evenly over ·3 A., we find from the curves of fig. 2 (and others interpolated between them),

for	$n =$	$10^{-4\cdot8}$	$10^{-5\cdot4}$	$10^{-6\cdot2}$	$10^{-6\cdot8}$;
	$r =$	·04	·08	·30	·48.

In all these cases the theoretical intensity in the centre is $< \cdot 01$. For still lower concentrations, where the real width of the line is small and the wings have disappeared, the central intensity of the broadened line rises nearer to 1. The total absorption of the line, or the equivalent width, is not changed by this instrumental widening, and may now be used to determine n. These considerations show the importance of taking stellar spectra with a much greater resolving power for the purpose of an exact determination of line contours.

* *M.N.*, **88**, 548 (1928). † *M.N.*, **89**, 222 (1929).
‡ *Astroph. J.*, **71**, 221 (1930).

This explanation of course holds only for some of the observed lines in spectra. In the case of the hydrogen lines the black Doppler nucleus has a width of 0·5 to 1 A., and cannot be diffused by the same instrumental causes. Still Elvey, in his three-prism spectra,* finds central intensities of ·19 and ·14 for Sirius (Hβ and Hγ), ·22 and ·16 for α Lyræ, ·13 for Procyon, and for most stars much larger values. In the case of the Sun we can free ourselves from the rotational as well as (for a large part) the instrumental widening by taking the spectrum of one part of the disc only and making use of a large dispersion. For Hα Minnaert finds a central intensity ·15 ; and for the K line ·07—smaller than other observers have found (Unsöld gives ·46 for Hα, ·16 for K) ; this may be an indication that large errors, especially by stray light which is very difficult to remove, may spoil such measures. If we take Minnaert's results as right, we have here a discrepancy between theory and observation which cannot be explained by any of the causes mentioned above. The irregular figure of the contour of the solar K line, giving rise to the reversal K_2 and the double reversal K_3, may perhaps be considered as an indication that reversals of temperature, or more generally irregular deviations from the adopted temperature and pressure laws in the atmosphere, are responsible for the observed central intensity.

13. *Appendix. The Continuous Absorption in the Atmospheric Layers of a Star.*—The formulæ derived by Gaunt give the absorption exerted on a stream of radiation by the interaction of an electron and a positively charged nucleus. If the energy E before absorption was positive, we have the case of a collision of a free electron which by absorption increases its energy. If E before absorption was negative, the electron was bound to the nucleus and we have the case of ionization by absorption of radiation. The formulæ hold strictly for the case of both energies as well as ν approximating to 0. For the case of finite ν with $E \to 0$ (+ or −), as well as for the case of finite ν and E (negative) with final $E' = 0$, a development in series showed that only multiplying factors g had to be added, not much different from 1 (1·2, 1·1, 0·9, 0·8).

The formula for positive E (free-free transitions) can be applied easily to the case of hyperbolic orbits around ionized atoms, which then may be considered as nuclei with charge $Z = 1$. It is not at once applicable to electrons penetrating into the atom and during part of their orbit subject to the stronger attraction of a higher charged centre. Their contribution to the absorption will increase the amount computed in the first-named case by an unknown factor.

The absorption coefficient for the case of free electrons with energy E and velocity v, as given by Gaunt's formula (6.10), identical with the formula derived formerly by Kramers by means of the correspondence principle, is

$$a_0(E, \nu) = \frac{4\pi Z^2 \epsilon^6}{3\sqrt{3} hcm^2 v \nu^3}$$

* The slit width is not given, but probably it was not much above ·3 A. for *H*γ.

for one atom per unit density of electrons. We take Z = 1. The average of $1/v$ for a Maxwellian distribution of velocities is given by

$$\overline{1/v} = \sqrt{\frac{\overline{2m}}{\pi k\mathrm{T}}}.$$

For the electron density we introduce its value $\mathrm{P}/k\mathrm{T}$, where P is the electron pressure. Denoting by a the fraction of the total mass which is formed by the active atoms, and by F the unknown factor just alluded to, we find the mass-absorption coefficient by dividing by $m_{\mathrm{H}}\mu$($= [\cdot 221 - 24]\mu$) ; the result is

$$k_0(\nu) = \frac{4\sqrt{6\pi}}{9} \frac{\epsilon^6 \mathrm{P} \,.\, a \,.\, \mathrm{F}}{hc(mk\mathrm{T})^{3/2}\nu^3 m_{\mathrm{H}}\mu}.$$

Introducing numerical values for the constants for the case of $\mathrm{T} = 6000°$, $\lambda = 4000$ A., we find

$$k_0(\nu) = [\cdot 889 - 3]\mathrm{P}a\mathrm{F}/\mu.$$

The formula for negative E (bound electrons thrown out by the ionization process) as given by Gaunt (6.21) is

$$a_1(\mathrm{E}, \nu) = \frac{1}{n^2} \frac{16\pi^2}{3\sqrt{3}} \frac{\mathrm{Z}^2\epsilon^6}{ch^3\nu^3}\Delta\nu,$$

where

$$\Delta\nu = \frac{4\pi^2 m\mathrm{Z}^2\epsilon^4}{h^3 n^3}.$$

Here again the ionized atom may be considered as a nucleus with charge Z = 1. Introducing the values of the constants (again taking 4000 A. and 6000°) this amounts to $[\cdot 834 \cdot 16]/n^5$, and the mass-absorption coefficient from this source becomes

$$k_1(\mathrm{E}, \nu) = [\cdot 613 + 8]/n^5\mu.$$

If the resulting energy $\mathrm{E}' = 0$, this absorption would be concentrated in a number of separate lines ν_n (for $n = 1, 2, 3, 4 \ldots$), each of which collects the energy within a region $\Delta\nu$ (which contains the factor $2/n^3$, for small n to be replaced, after Milne, by $\frac{1}{(n - 1/2)^2} - \frac{1}{(n + 1/2)^2}$). For another E′ all these ν_n are replaced by somewhat larger values, $\nu = \nu_n + \mathrm{E}'/h$; thus each ν_n is extended into a continuous band, reaching from ν_n as its lower sharp edge towards ∞. To compute the absorption cœfficient for a certain ν we have to add the contributions of all bands which have their edges at a lower $\nu_n < \nu$; the strength of each band depends on the relative number of atoms in the corresponding state, for which we take the Boltzmann formula ; hence

$$a_1(\nu) = \frac{1}{\nu^3} \frac{16\pi^2\epsilon^6}{3\sqrt{3}ch^3} \sum_{(n)}^{\infty} \frac{1}{n^2} e^{-h(\nu_1 - \nu_n)/k\mathrm{T}}\Delta\nu.$$

This value behaves very irregularly (*vide* fig. 4), decreasing slowly and then jumping up to a higher value at each band edge. In order to have an expression independent of the chance values of the excitation energies, we will for the summation substitute an integral, assuming a quasi-infinite number of band edges, each of which has $d\nu$ instead of the finite $\Delta\nu$. In the simplified case represented by the Gaunt formula we may put $n^2/1 = h\nu_1/h\nu_n$; then the integral becomes

$$\int_0^{\nu} \frac{\nu_n}{\nu_1} e^{-h(\nu_1-\nu_n)/kT} d\nu_n = \frac{kT}{h}\left\{\left(\frac{\nu}{\nu_1} - \frac{kT}{h\nu_1}\right) e^{-h(\nu_1-\nu)/kT} + \frac{kT}{h\nu_1} e^{-h\nu_1/kT}\right\}$$

The last term is small compared with the others, and may be omitted. The result is that the factor $\Delta\nu(= [\cdot 818 + 15]Z^2/n^3)$ is replaced by $\frac{kT}{h}(= [\cdot 099 + 14]$, so that the constant in k_1 becomes $[6\cdot 883])$, multiplied by a mainly exponential factor depending on the ionization

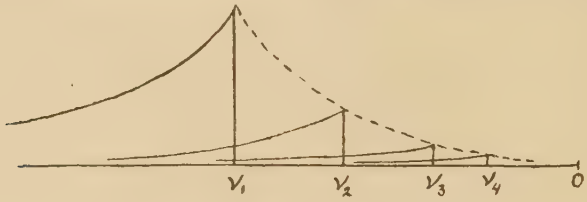

FIG. 4.

potential. We can express all the energies by corresponding potentials ($h\nu = \epsilon V/300$, $kT = \epsilon V_T/300$); then the expression above becomes

$$\frac{V - V_T}{V_1} e^{(V_1 - V)/V_T}.$$

Here V_1 is the ionization potential of the element. For wave-length 4000 A., $V = 3\cdot 1$; for $T = 6000°$, $V_T = 0\cdot 52$; the absorption coefficient per atom in this case will be

$$[6\cdot 883]\frac{2\cdot 6}{V_1} e^{-(V_1 - 3\cdot 1)/\cdot 52}.$$

For atoms with a high ionization potential this exponential form is very small; only atoms with a low ionization potential will give an appreciable contribution to this absorption coefficient. These atoms, however, are to a large part ionized, and only a small neutral fraction is able to take part in this absorbing process. From the ionization formula, expressed in the quantities above, viz.

$$\frac{x_1}{x_0} P = 10^{-0\cdot 5} T^{+2\cdot 5} e^{-V_1/V_T} \qquad (P \text{ in absolute units})$$

we derive for the cases of strong ionization, when x_1 is nearly 1,

$$x_0 = 10^{0\cdot 5} T^{-2\cdot 5} e^{+V_1/V_T} P.$$

If we multiply the computed absorption coefficient by this fraction we get the average absorption coefficient per atom of this element present in the atmosphere (all states of ionization counted together). In this product

$$[7{\cdot}38]\frac{V - V_T}{V_1}T^{-2{\cdot}5}e^{V/V_T}P,$$

the ionization potential has disappeared (but for the unimportant factor $(V - V_T)/V_1$), whereas the electron pressure appears as a factor of this part of the absorption coefficient too. Hence this expression is the same for the different elements of low ionization potential and each contributes according to its concentration in the stellar atmosphere. This holds as long as the ionized fraction remains above, say, 1/2 (for $T = 6000°$ and $P = 10^{-4}$ atmospheres this corresponds to an ionization potential of 8 and comprises *K*, *Na*, *Ca*, *Sr*, *Ti*, *Cr*, perhaps *Fe*); for other elements the neutral fraction increases very slowly towards 1, and the product of this fraction with the absorption coefficient decreases rapidly with increasing ionization potential.

Restoring the analytical expressions for $a_1(\nu)$ and for $\frac{x_1}{x_0}$, *viz.*

$$a_1(\nu) = \frac{16\pi^2}{3\sqrt{3}}\frac{\epsilon^6}{ch^3\nu^3}\frac{kT}{h}\left(\frac{\nu}{\nu_1} - \frac{kT}{h\nu_1}\right)e^{h\nu/kT - h\nu_1/kT}$$

$$\frac{x_1}{x_0}P = \frac{(2\pi m)^{3/2}}{h^3}(kT)^{5/2}e^{-h\nu_1/kT},$$

multiplying with a factor a' (denoting the concentration of the relevant elements), and dividing by $m_H\,\mu$, to have the mass-absorption coefficient we find

$$k_1(\nu) = \frac{4\sqrt{6\pi}}{9}\frac{\epsilon^6 P a'}{hc(mkT)^{3/2}\nu^3 m_H\mu}\left(\frac{\nu}{\nu_1} - \frac{kT}{h\nu_1}\right)e^{h\nu/kT}.$$

This expression is the same as our expression for $k_0(\nu)$, only with the addition of two factors at the end. In both a factor indicating the concentration of relevant atoms occurs. The factor a in $k_0(\nu)$ is the fraction of the atoms ionized; the most numerous atoms, those of hydrogen (and helium) are not ionized, and among the less numerous metal atoms we may suppose half of them ionized. The fraction a' in $k_1(\nu)$ indicates the abundance of atoms of those elements that are at least half ionized. These fractions will be of the same order of magnitude. The additional factor in $k_1(\nu)$ amounts to $[2{\cdot}59 - 0{\cdot}4]$; hence it appears that the ionization part of the absorption coefficient comes out 150 times larger than the collision part, at least if the latter, by the unknown factor F introduced in the beginning, is not increased in the same ratio. If this is not the case, we may to a first approximation neglect the collision part in comparison with the other one. Introducing numerical values we have then

$$k_1(\nu) = [{\cdot}89 - 3][2{\cdot}59 - 0{\cdot}4]\frac{Pa'}{\mu}.$$

As to the concentration a' we can at present only make a rough guess. We will assume that the metal atoms are 1 per cent. of the total mass, and that one half of them contribute to the formula. Hence taking $a' = 1/200$, and μ, their mean atomic weight $= 40$, we find

$$k(\nu) = \kappa_0 \mathrm{P} = [-\ 3{\cdot}8]\mathrm{P}.$$

This value has been used in the computations of § 9. Probably, however, it is too low, because the unknown factor F in k_0 is underrated here. We can indicate an upper limit for it. Treating the electrons penetrating into the atoms as meeting a single nucleus with $Z = 20$, the coefficient of absorption from free-free transitions would become 400 times larger. Still more important is the contribution of the hydrogen atoms, if we may treat the electrons penetrating into them as if they meet single nuclei with charge $Z = 1$ and $\mu = 1$. Because their abundance is nearly 1, they would produce in this case a coefficient $200 \times 40 = 8000$ times higher than the ionized metal atoms do. This is an upper limit for F ; it means that the total k, in which in this case free-free transitions play the dominant part, may be at most 50 times the value given above, *i.e.* $\kappa_0 < [-\ 2{\cdot}1]$. The great abundance of hydrogen atoms in stellar atmospheres makes it highly desirable that the absorption caused by free-free transitions of electrons along the nucleus of a neutral hydrogen atom should be investigated more exactly ; this appears to be the chief agent determining the transparency of such atmospheres. Since we cannot make as yet any estimate of its real value, we have not changed the above value in the computations of § 9.

14. We can also use these results to derive the mean absorption coefficient of such an atmosphere for the total radiation. The net stream of all wave-lengths which passes through the atmosphere is a given quantity ; hence the net stream equation for a single ν :

$$\frac{dy(\nu)}{\rho dh} = k(\nu)z \quad \text{or} \quad \frac{1}{k(\nu)}\frac{dy(\nu)}{\rho dh} = z(\nu)$$

has to be written for the total radiation

$$\sum_\nu z(\nu) = \sum_\nu \frac{1}{k(\nu)}\frac{dy(\nu)}{\rho dh} = \frac{\bar{1}}{k(\nu)}\frac{d}{\rho dh}\sum y(\nu).$$

It is not the simple mean of the absorption coefficients which we want, but their harmonic mean ; it is not the absorption but the transparency that is averaged.

In the case of an absorption coefficient produced by free-free transitions the derivation of this $k = 1/\frac{\bar{1}}{k(\nu)}$ is easy. The only variable factor in $k(\nu)$ is $1/\nu^3$; and we have

$$\int \nu^3 \mathrm{E}_\nu d\nu \Big/ \int \mathrm{E}\ d\nu = \int \frac{\nu^6 d\nu}{e^{h\nu/k\mathrm{T}} - 1} \Big/ \int \frac{\nu^3 d\nu}{e^{h\nu/k\mathrm{T}} - 1} = \left(\frac{k\mathrm{T}}{h}\right)^3 \int \frac{x^6 dx}{e^x - 1} \Big/ \int \frac{x^3 dx}{e^x - 1}.$$

The second integral is $\frac{1}{1}\frac{1}{5}\pi^4 = 6{\cdot}49$; the first can be written

$$6!(1 + 2^{-7} + 3^{-7} + \ldots) = 726{\cdot}01 ;$$

so that instead of $1/\nu^3$ in the $k(\nu)$ we have $\left(\frac{h}{kT}\right)^3 \frac{6{\cdot}49}{726{\cdot}01}$ in the average. To obtain the general absorption coefficient we have to multiply the coefficient, computed for ν, by the factor

$$\frac{1}{111{\cdot}80}\left(\frac{h\nu}{kT}\right)^3.$$

In our case ($T = 6000°$, ν for 4000 A.) $h\nu/kT = 5{\cdot}97$ and the factor amounts to $213/112 = 1{\cdot}91$. Milne, in computing the simple mean, had to multiply $k(\nu)$ by $\frac{15}{\pi^4}\left(\frac{h\nu}{kT}\right)^3$, which gives a result 17 times greater than the harmonic mean.

In the case of $k(\nu)$ produced by ionization the matter is much more difficult. Here a number of bands are superposed, each with a sharp edge at the lower side, so that the transparency is largest just below these edges. It is clear that the average transparency will depend in a high degree on the number and the distribution of these bands, *i.e.* on the different ionization potentials. The procedure of smoothing them into an infinite number of band edges cannot be followed here, because then just the most transparent parts would be obliterated. In the formula

$$\text{harm. mean } k = \int E(\nu)d\nu \Big/ \int \frac{E(\nu)}{k(\nu)}d\nu$$

the latter integral breaks up into a number of separate terms, between the limits ν_1, ν_2, ν_3 . . ., the ionization band edges. We omit first the common factor in $k(\nu)$,

$$\frac{64\pi^4 Z^4 \epsilon^{10} m}{3\sqrt{3}ch^6} \quad \text{(in which Z shall be taken = 1)},$$

and preserve only the variable factor $\frac{e^{-h(\nu_1-\nu_n)/kT}}{\nu^3 n^5}$. In the approximation of Gaunt's deduction, $1/n^2$ represents $h\nu_n/h\nu_1$, so that n^5 in these integrals may be replaced by 1, $(\nu_1/\nu_2)^{5/2}$, $(\nu_1/\nu_3)^{5/2}$, etc. If it is permissible to neglect in each interval between two edges all the more remote fainter bands and to preserve only the next one, we have for this latter integral

$$\int_{\nu_1}^{\infty} E\nu^3 d\nu + \left(\frac{\nu_1}{\nu_2}\right)^{5/2} e^{(h\nu_1-\nu_2)/kT}\int_{\nu_2}^{\nu_1} E\nu^3 d\nu + \left(\frac{\nu_1}{\nu_3}\right)^{5/2} e^{(h\nu_1-\nu_3)/kT}\int_{\nu_3}^{\nu_2} E\nu^3 d\nu + \ldots$$

The arbitrary character of this value in consequence of our ignorance as to the chance values of these separate ν_n can be avoided in the following way :—

If the discontinuous factors $\left(\frac{1}{\nu_n}\right)^{5/2} e^{-h\nu_n/kT}$ before the integrals are

replaced by the continuously changing values $\left(\frac{1}{\nu}\right)^{5/2} e^{-h\nu/kT}$, the integration can be extended from 0 to ∞ without knowledge of the band edges. The integral is then too small because for the integrand is taken a continuous curve through the minimum values at the darkest edges, and the real values from these points rapidly increase to the transparent parts close to the next edges. For the factor f, by which it has to be multiplied, we take the mean ratio between the real, increasing values and the minimum edge value within each interval. The minimum integral is given by

$$\nu_1^{5/2} e^{h\nu_1/kT} \int_0^\infty \frac{\nu^{7/2}}{e^{h\nu/kT}-1} e^{-h\nu/kT} d\nu = \nu_1^{5/2} e^{h\nu_1/kT} \left(\frac{kT}{h}\right)^{9/2} \int_0^\infty \frac{x^{7/2}}{e^x - 1} e^{-x} dx$$

for which the numerical value

$$\cdot 64 \nu_1^{5/2} e^{h\nu_1/kT} \left(\frac{kT}{h}\right)^{9/2}$$

can be found. For the factor f, by which it has to be multiplied,

$$f = \frac{1}{x_n - x_{n+1}} \int_{x^n+1}^{x_n} \left(\frac{x}{x_{n+1}}\right)^{5/2} e^{x - x_{n+1}} dx,$$

cannot be given an exact value, because it depends on the unknown quantities x_n and $\Delta x = x_n - x_{n+1}$. To avoid mathematical difficulties it was evaluated, after substituting the exponents 3 and 2 for 5/2, for different values of x_n and Δx. Since in the minimum integral only the contributions of x between $x = 1$ and $x = 5$ are relevant, we want the factor f only for x_n between these limits. Here f increases from nearly 1·2 to nearly 8, if Δx increases from ·5 to 2. It is not probable that for such x_n intervals as large as 2 in the values of $h\nu/k\mathrm{T}$ between successive band edges will occur, especially because a number of elements with ionization potential below 7 or 8 take part in the absorption, and each produces a number of band edges. It is not probable, therefore, that the uncertainty reaches this factor 8, though it is difficult to ascertain a trustworthy lower limit.

For the harmonic mean of k, without the constant factors, and without the concentration factor for the neutral atoms taking part in the process, we now have

$$\frac{\pi^4}{15}\left(\frac{kT}{h}\right)^4 \Big/ \cdot 64 f \cdot \left(\frac{h\nu_1}{kT}\right)^{5/2} \left(\frac{kT}{h}\right)^7 e^{h\nu_1/kT} = \frac{6\cdot 49}{\cdot 64 f}\left(\frac{h}{kT}\right)^3 \left(\frac{h\nu_1}{kT}\right)^{-5/2} e^{-h\nu_1/kT}.$$

The non-ionized fraction of the atoms is given by

$$x_0 = (2\pi m)^{-3/2} h^3 (kT)^{-5/2} e^{h\nu_1/kT} \mathrm{P},$$

and with a' again denoting the abundance of the atoms concerned and μ_a their atomic weight, the total formula for the continuous absorption is found :

$$k = \frac{16\pi^6\sqrt{2\pi}}{45\sqrt{3}} \cdot \frac{\epsilon^{10}}{cm^{\frac{1}{2}} m_{\mathrm{H}}} (kT)^{-5\frac{1}{2}} \left(\frac{kT}{h\nu_1}\right)^{5/2} \frac{1}{\cdot 64 f} \cdot \frac{a'}{\mu} \mathrm{P}.$$

Introducing numerical values, taking $h\nu_1/k\mathrm{T} = 6{\cdot}1$ Volt/$\cdot 52$ Volt $= 12$ for the average atoms, and assuming as above $a' = 1/200$, $\mu_a = 40$, we find $k = [-3{\cdot}7]\frac{1}{f}\mathrm{P}$.

In this formula the coefficient is proportional to the $-5\frac{1}{2}$th power of the temperature, whereas in Kramers's formula the power is $-4\frac{1}{2}$. There is, however, an additional factor containing T. Now with increasing temperature, more metals with higher ionization potentials take a perceptible part in this absorption; hence the average ν_1 increases with T, and the factor $k\mathrm{T}/h\nu_1$ certainly is not proportional to T. It may, however, remove part of the power $-5\frac{1}{2}$ in the preceding factor.

The Resonance Theory of the Origin of the Moon (*Second Paper*).

By Harold Jeffreys.

1. It was shown by Sir G. H. Darwin that if the earth and moon were formerly united as a single fluid mass with their present total angular momentum, the period of rotation would be about 4 hours, so that the period of the semi-diurnal tide raised by the sun at any given place would be about 2 hours. This is approximately the period of the slowest free oscillation of a fluid mass of the earth's mean density, and Darwin suggested in consequence that the sun's tides would have been enormously magnified by resonance. The principle is that if a system is able to oscillate, and disturbing forces are so timed that in whatever direction their point of application is moving of its own accord we give it a push in the direction of motion, an indefinitely large amplitude may be ultimately worked up. If the mass became drawn out into a sufficiently long ellipsoid it would become unstable, and the suggestion was that at this stage a portion broke off to form the moon.

It has always been recognized that the validity of this theory depends on the smallness of friction; for friction necessarily fixes an upper limit to the amplitude of an oscillation attainable by resonance. In an earlier paper * I showed that the requisite coincidence of periods can be attained only if we allow for the earth's lack of homogeneity; the central core must be considered. This introduces a possibility of friction in the tidal currents in the outer shell flowing over the central core. This friction can now be estimated; and it turns out to be sufficient to invalidate the theory.

2. A complete solution of the problem would be very laborious, but fortunately turns out to be unnecessary. We notice first that if we have a system specified by a differential equation of the type

$$\ddot{x} + n^2 x = f \sin nt \quad . \quad . \quad . \quad . \quad (1)$$

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* *M.N.R.A.S.*, **78**, 116–131, 1917.

where $2\pi/n$ is the free period, x the disturbed co-ordinate, and f a constant, and the system starts from rest, the solution is

$$x = \frac{f}{2n^2}(\sin nt - nt\cos nt) \quad . \quad . \quad . \quad (2)$$

The conventional " equilibrium displacement " determined by ignoring $\ddot{x}$ in (1) is

$$x_0 = \frac{f}{n^2}\sin nt \quad . \quad . \quad . \quad . \quad (3)$$

Then the first term in (2) is simply half the equilibrium displacement. The second represents an oscillation whose amplitude increases steadily with the time. The increase of amplitude in a period $2\pi/n$ is equal to π times the amplitude of the equilibrium displacement, f/n^2. The phase is always $\frac{1}{2}\pi$ behind that of the disturbing force. If the term in the kinetic energy corresponding to this degree of freedom is $\frac{1}{2}m\dot{x}^2$, the work done by the disturbance in a period $2\pi/n$ is

$$\int_{2r\pi/n}^{2(r+1)\pi/n} mf\sin nt\,.\,\dot{x}dt = \tfrac{1}{2}\int mf^2t\sin^2 nt\,dt = \frac{\pi}{2n}mf^2t\,. \quad (4)$$

where t is evaluated at the middle of the interval. The mean rate of increase of energy is therefore

$$\frac{d\mathrm{E}}{dt} = \tfrac{1}{4}mf^2t \quad . \quad . \quad . \quad . \quad . \quad (5)$$

But the potential energy for a displacement x is $\frac{1}{2}mn^2x^2$, and if the system had always its equilibrium displacement its mean potential energy would be $\frac{1}{4}mn^2\left(\frac{f}{n^2}\right)^2$ or $\frac{1}{4}mf^2/n^2$. Thus the mean rate of increase of energy is

$$\frac{d\mathrm{E}}{dt} = n^2t\mathrm{V}_0 \quad . \quad . \quad . \quad . \quad (6)$$

where V_0 is the mean potential energy of the equilibrium disturbance. All quantities in this equation are easily calculated.

3. In our problem the tidal elevation at any place when considerable magnification has been attained will occur a quarter of a period after the place has passed through the plane that includes the sun and the earth's axis, so that the tides are apparently carried forward through an angle of 45°. A section by the plane of the equator will be as in fig. 1.

The earth rotating counter-clockwise in the figure, the greatest tidal elevations will be at A and C, and the greatest depressions at B and D. The diameter COA is being steadily extended, keeping at a fixed angle 45° to the line joining O to the sun. It therefore revolves in a year, while the vorticity is uniform and equal to 2ω as for a body

rotating as if rigid in the actual period. The motion of the boundary is therefore very slow in comparison with the rotation, and for the purpose of estimating the velocities we may consider the fluid as rotating with constant vorticity within a fixed envelope. The motion therefore consists of a rotation with angular velocity ω as for a rigid body, with such an irrotational motion superposed on it as will keep the velocity at the boundary everywhere tangential.

Instability will arise, if at all, when the radius OA is about twice the mean radius. If we define the mean radius a by the condition that the volume is $\frac{4}{3}\pi a^3$, the radius OA is about $2a$, and OB about $a/\sqrt{2}$ (the radius along the axis of rotation being supposed reduced in the same ratio as OB). The radius of the core is about $\frac{1}{2}a$, as we now know

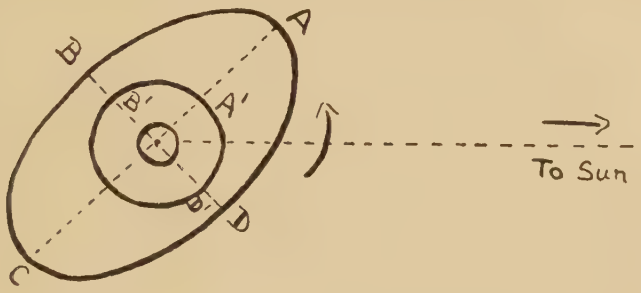


Fig. 1

from seismology. We shall at present consider the core as undeformed. Then the fluid passing through the gap of thickness $a\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)$ below B in the period of rotation also passes through the gap of thickness $2a - \frac{1}{2}a$ below A. The mean velocity across the gap below B is therefore $3(\sqrt{2}+1)$, or say seven times that at A. If we take the core as rotating uniformly, the fluid of the shell at B′ is moving several times faster than the core; that at A′ is hardly moving at all. The essential point is that the velocity of slip at B′ is forwards, and at A′ backwards, the range being of order $\frac{1}{2}a\omega$ on the equator. We may allow for the effect of latitude by introducing a factor $\sin\theta$, where θ is the colatitude.

Now fluid flowing over a boundary ordinarily produces a frictional force about equal to $0{\cdot}002\rho u^2$ per unit area, where u is the velocity of slip, ρ the density, and 0·002 an empirical numerical coefficient. This is applicable in most phenomena of fluid flow on the large scale; the excluded phenomena are on the scale of flow in capillary tubes, lubrication, and the fall of mist drops. There is therefore a dissipation of energy $0{\cdot}002\rho u^3$ per unit area per unit time, where the velocity is always to be taken without regard to sign. In our problem we must

integrate over the surface of the core ; then F, the rate of dissipation of energy, is of order given by

$$\mathrm{F} = \int_0^{\pi}\int_0^{2\pi} 0{\cdot}002\rho(\tfrac{1}{2}a\omega\sin\theta)^3(\tfrac{1}{2}a)^2\sin\theta\,d\theta d\phi$$

$$= 5\times 10^{-4}\rho a^5\omega^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The total energy is greater than for a sphere. The excess due to the deformation, involving velocities of the same order as those due to pure rotation, must be of the same order as the kinetic energy of a sphere rotating with angular velocity ω, and can be taken to be

$$\mathrm{T} = \tfrac{1}{2}\mathrm{C}\omega^2 \quad . \quad . \quad . \quad . \quad (2)$$

where C is the earth's moment of inertia. If ρ is the mean density of the earth and M its mass,

$$\mathrm{C} = \tfrac{1}{3}\mathrm{M}a^2\,;\ \mathrm{M} = \tfrac{4}{3}\pi\rho a^3 \quad . \quad . \quad . \quad (3)$$

$$\mathrm{T} = \tfrac{2}{3}\rho a^5\omega^2 \quad . \quad . \quad . \quad . \quad (4)$$

roughly. The dissipation would annihilate this in a time

$$\frac{\mathrm{T}}{\mathrm{F}} = \frac{1{\cdot}3\times 10^3}{\omega} \quad . \quad . \quad . \quad . \quad (5)$$

For rotation in four hours,

$$\omega = 2\pi/(4\text{ hours})\,;\qquad \mathrm{T/F} = 800\text{ hours} = 30\text{ days}\,. \quad (6)$$

roughly. For resonance the supply of energy from the sun's tidal action must be enough to overcome this dissipation.

4. Now the equilibrium amplitude of the solar tide being taken as b at its maximum, b is about 10 cm. The elevation at a given place can be taken as $b\sin^2\theta\cos 2\phi$, with a suitable zero of longitude ; and the potential energy per unit area of the surface is $\frac{1}{2}g\rho$ times the square of this. Hence the total potential energy of the equilibrium disturbance is

$$\mathrm{V}_0 = \tfrac{1}{2}g\rho\int_0^{\pi}\int_0^{2\pi} b^2(\sin^2\theta\cos 2\phi)^2a^2\sin\theta\,d\theta d\phi$$

$$= 2g\rho a^2b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

roughly. In our problem the magnification required is of order

$$a/b = \tfrac{1}{2}nt \quad . \quad . \quad . \quad . \quad (2)$$

by comparing 2 (2) and 2 (3) ; and the rate of supply of energy, by 2 (6), is

$$\frac{2a}{b}n\mathrm{V}_0 = 8g\rho a^3b\omega \quad . \quad . \quad . \quad . \quad (3)$$

the free period being $2\pi/2\omega$. The ratio of this rate to the rate of dissipation F is practically

$$\lambda = 1{\cdot}6 \times 10^4 \frac{gb}{a^2\omega^2} . \quad . \quad . \quad . \quad (4)$$

Now at present $g/a\omega^2$ is about 290; for an earth rotating 6 times as fast it would be about 8. Taking b as 10 cm., a as 6×10^8 cm., we have

$$\lambda = 2 \times 10^{-3}.$$

If the requisite magnification were ever attained, the friction would dissipate the energy 500 times as fast as the sun's tidal pull supplied it. This inconsistency implies that the tidal deformation could never attain the requisite magnitude, and the resonance theory accordingly fails.

It is of some interest to estimate the height actually attainable. The rate of dissipation is proportional to the cube of the actual amplitude, and the rate of supply to the product of the equilibrium and the actual amplitudes. The equilibrium amplitude being given, the ratio is proportional to the square of the actual amplitude. Thus λ could be raised to unity by dividing the amplitude considered by 500 or 23. The elevation would then be about $\frac{1}{23}$ of the radius and there could be no question of instability.

5. We assumed earlier that the core would remain spherical. To test this postulate we notice that the surface of the core is a stream surface. Along any stream line

$$\frac{p}{\rho} + \tfrac{1}{2}q^2 - \mathrm{U} = \text{constant},$$

where p is the pressure, q the resultant velocity, and U the gravitation potential. At A′ in the figure the velocity is less and the potential greater than at B′; hence the pressure is greater at B′, and the core will be pushed inwards there. This intensifies the difference between the depths of the shell below A and B and strengthens the argument.

6. The abandonment of the resonance theory seems to drive us to the opinion that the moon has been separate from the earth ever since the early catastrophe that formed the planets. On the tidal theory of the origin of the solar system the majority of the satellites were supposed to be formed by the disruptive action of the solar tides in their primaries at the first perihelion passage, but this method appeared inapplicable to the moon on account of the earth's high density. The resonance theory therefore seemed the only reasonable alternative. It appears now, however, that there are strong reasons for believing that the original disruptive encounter of the sun and star was not merely a close approach but an actual collision. The formation of satellites on this theory is not clearly understood as yet, but at all events it seems unlikely that the moon will offer any difficulty not presented by any other satellite.

A Note on the Possible Emission of Electric Currents from the Sun.

By V. C. A. Ferraro.

(*Communicated by Professor Chapman.*)

1. In connection with terrestrial magnetic storms it is of interest to examine whether electric currents can be maintained in an ionized stream of corpuscles continuously emitted from the Sun's atmosphere by radiation-pressure. The question arose in the course of discussions with Professor S. Chapman, and at his suggestion this investigation was undertaken.

The problem which it is desirable to consider is the emission of ions from a *restricted* region on the Sun's surface, but great mathematical difficulties arise in this case, and it has been found necessary to simplify the problem: the principal modification made is to assume that the ions are emitted from all over the Sun, but not equally in all directions; thus one of the main features of the original problem is retained. As we shall see below, the conclusion deduced in this simple problem seems likely to hold in more general cases.

2. According to Professor Milne's theory of emission of high-speed atoms from the Sun,* radiation-pressure acts continuously and radially outwards on the charges, but primarily on the positive ions; the acceleration of the electrons due to radiation-pressure can, in fact, be neglected.

As soon as some positive ions have been emitted they will drag after them, by electric forces, a nearly equal number of free electrons from the Sun's atmosphere. It does not seem necessary to consider in detail the initial stages of the emission of the stream, since it will be possible to decide from the solution for the steady state whether any appreciable currents can be maintained within the stream.

In the absence of any disturbance, radiation-pressure will support the ionized atmosphere against gravity; this, therefore, need not be considered beyond the fact that it causes a small electrostatic field keeping together the electrons with the positive ions in the atmosphere. The resistance due to motion (diffusion) of one set of charges through the set of opposite sign can also be neglected.

We shall suppose that the radial outward acceleration, **I**, produced by radiation-pressure on a positive ion, is given by

$$\mathrm{I} = \frac{\mathrm{I}_0 + \mathrm{I}_1 \mathrm{P}_1(\cos\theta)}{r^2} \quad . \quad . \quad . \quad (1)$$

where r is the radial distance of a point from the Sun's centre, θ, the angular distance of the point from some fixed solar radius, and I_0, I_1 are positive constants. Since the acceleration is outwards, I is positive or zero, and hence $\mathrm{I}_1 \not> \mathrm{I}_0$. If we put $\mathrm{I}_1 = 0$ in (1), we obtain Professor

* E. A. Milne, *M.N.R.A.S.*, **86**, 459, 1926.

Milne's expression for the acceleration of an ion after it has climbed out of its absorption line.*

3. Let the suffixes i and e apply to the positive ions and electrons respectively, and denote (with appropriate suffix i or e) the mean velocity of either set of charges by $\mathbf{V}$, their mean number density by n, the mass of an individual charge by m, the force per unit mass acting on the charges by $\mathbf{F}$. In the steady state the equations of motion and continuity of either set of charges are respectively:

$$-2\mathbf{V} \wedge \omega = \mathbf{F} - \frac{1}{2}\frac{\partial}{\partial \mathbf{r}}\mathbf{V}^2 \quad . \quad . \quad . \quad (2)$$

$$div\,(n\mathbf{V}) = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ω denotes the vorticity, given by the relation

$$\omega = \tfrac{1}{2}\,curl\,\mathbf{V} \quad . \quad . \quad . \quad . \quad (4)$$

Let $\mathbf{E}$ denote the electric field in e.s.u., $\mathbf{H}$ the magnetic field in e.m.u. $+e$ and $-e$ the charge carried respectively by an ion and an electron in e.s.u., and c the velocity of light. Considering an element of volume dv of, say, the electron atmosphere in the stream, containing $n_e dv$ electrons, the force per unit volume acting on this element is

$$-n_e e\left(\mathbf{E} + \frac{1}{c}\mathbf{V}_e \wedge \mathbf{H}\right);$$

by definition this is also equal to $n_e m_e \mathbf{F}_e$. Hence, we have

$$\mathbf{F}_e = -\frac{e}{m_e}\mathbf{E} - \frac{e}{m_e c}\mathbf{V}_e \wedge \mathbf{H} \quad . \quad . \quad . \quad (5)$$

and, similarly,

$$\mathbf{F}_i = \mathbf{I} + \frac{e}{m_i}\mathbf{E} + \frac{e}{m_i c}\mathbf{V}_i \wedge \mathbf{H} \quad . \quad . \quad . \quad (6)$$

We note that both these expressions are independent of the number density of the corpuscles.

4. In the steady state, $\mathbf{E}$ will be derived from a potential Φ which is determined from a knowledge of the distribution of electric charges in the stream and its induced charges on the Sun. Thus $E = -\partial\Phi/\partial\mathbf{r}$, and writing further

$$\left.\begin{aligned} \mathbf{J} &= \frac{e}{m_e c}\mathbf{H} - 2\omega_e \\ \mathbf{J}' &= \frac{e}{m_i c}\mathbf{H} + 2\omega_i \\ \Omega &= \frac{1}{2}\mathbf{V}_e^2 - \frac{e}{m_e}\Phi \\ \Omega' &= \frac{1}{2}\mathbf{V}_i^2 + \frac{e}{m_i}\Phi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (7)$$

* *Loc. cit.*, p. 464.

the equations of motion of the electrons and positive ions become respectively :

$$\mathbf{V}_e \wedge \mathbf{J} + \frac{\partial \Omega}{\partial \mathbf{r}} = 0 \quad . \quad . \quad . \quad (8)$$

$$-\mathbf{V}_i \wedge \mathbf{J}' + \frac{\partial \Omega'}{\partial \mathbf{r}} = \mathbf{I} \quad . \quad . \quad . \quad (9)$$

To render these equations tractable it is necessary to make an approximation, viz. that I_1/I_0 in (1) is a small quantity. It is then possible to write the densities and velocities of the corpuscles in either of the two simple forms

$$A + B\cos\theta, \qquad C\sin\theta \quad . \quad . \quad . \quad (10)$$

where A, B, C are functions of the distance r, and B/A, C/A are small quantities such that their squares and products may be neglected.

Choosing a system of spherical polar co-ordinates (r, θ, ϕ) with origin at the centre of the Sun, the θ-axis being the same as the polar axis in (1), and writing (U, V, 0) for the components of $\mathbf{V}$ in this system of axes, the following are found to be the expressions for the velocities and densities of the charges :—

$$\left.\begin{aligned} n_e &= \nu_0(r) + \nu_1(r)\cos\theta \\ n_i &= \nu_0'(r) + \nu_1'(r)\cos\theta \\ U_e &= u_0(r) + u_1(r)\cos\theta \\ V_e &= \quad\quad v_1(r)\sin\theta \\ U_i &= u_0'(r) + u_1'(r)\cos\theta \\ V_i &= \quad\quad v_1'(r)\sin\theta \end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

5. The volume charge-density at any point of the stream is

$$\rho = (n_i - n_e)e,$$

or

$$\rho = \rho_0 + \rho_1\cos\theta \quad . \quad . \quad . \quad . \quad (12)$$

where

$$\left.\begin{aligned} \rho_0 &= (\nu_0' - \nu_0)e \\ \rho_1 &= (\nu_1' - \nu_1)e \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (13)$$

If Φ_a is the potential at the Sun's surface ($r = a$, say), then considering the Sun to be an ordinary conductor, the potential Φ at any point outside the Sun's surface is found to be

$$\Phi = \Phi_0 + \Phi_1\cos\theta \quad . \quad . \quad . \quad (14)$$

where

$$\left.\begin{aligned} \Phi_0 &= \Phi_a\left(\frac{a}{r}\right) + \frac{4\pi}{r}\int_a^r \rho_0 r^2 dr \\ \Phi_1 &= \frac{4\pi}{3}\left\{ r\int_r^\infty \rho_1 dr + \frac{1}{r^2}\int_a^r \rho_1 r^3 dr - \frac{a^3}{r^2}\int_a^\infty \rho_1 dr \right\} \end{aligned}\right| \quad . \quad (15)$$

the stream being supposed to extend to infinity in all directions.

The value of Φ_a is to be determined from the condition that the sum-total of the charges induced on the Sun and the charges within the stream is zero.

6. The magnetic field produced by the currents in the stream is given by the relation

$$\mathbf{H} = curl\ \mathbf{A},$$

where $\mathbf{A}$ is the magnetic vector potential; at any point P within the stream we have

$$\mathbf{A}_P = \int \frac{\mathbf{i}}{r} dv \quad . \quad . \quad . \quad . \quad (16)$$

the integral being taken throughout the volume of the stream, and r denoting the distance of the element of volume dv from P.

The current density $\mathbf{i} = (e/c)(n_i\mathbf{V}_i - n_e\mathbf{V}_e)$ e.m.u., and neglecting second order quantities we have from (11)

$$\mathbf{i} = (i_0 + i_1 \cos\theta,\ i_1' \sin\theta,\ 0) \quad . \quad . \quad . \quad (17)$$

where

$$\left.\begin{aligned} i_0 &= (e/c)(\nu_0' u_0' - \nu_0 u_0) \\ i_1 &= (e/c)(\nu_0' u_1' + \nu_1' u_0' - \nu_0 u_1 - \nu_1 u_0) \\ i_1' &= (e/c)(\nu_0' v_1' - \nu_0 v_1) \end{aligned}\right\} \quad . \quad . \quad (18)$$

From (16) and (17), it is clear that $\mathbf{A}$ has components

$$(A_1,\ A_2,\ 0),$$

where A_1, A_2 are certain integrals which need not be considered in detail; since $\mathbf{A}$ is independent of the angle ϕ, we find

$$\mathbf{H} = curl\ \mathbf{A} = (0,\ 0,\ H_\phi), \quad . \quad . \quad . \quad (19)$$

where

$$H_\phi = \frac{1}{r}\left(\frac{\partial r A_2}{\partial r} - \frac{\partial A_1}{\partial \theta}\right)$$

The value of H_ϕ is readily obtained from the equation

$$curl\ \mathbf{H} = 4\pi\mathbf{i}.$$

From (17) and (19) this equation has the following components:—

$$\left(\frac{1}{r\sin\theta}\frac{\partial H_\phi \sin\theta}{\partial\theta},\quad -\frac{\partial r H_\phi}{\partial r},\quad 0\right) = 4\pi(i_0 + i_1\cos\theta,\quad i_1'\sin\theta,\quad 0),$$

the θ-component of which can be integrated once with respect to r; we find

$$H_\phi = H(r)\sin\theta + \text{const}/r,$$

where $H(r)$ is a function of r vanishing with i_1'. Since $\mathbf{H}$ vanishes with $\mathbf{i}$, in the present case the constant is zero and

$$H_\phi = H(r)\sin\theta \quad . \quad . \quad . \quad . \quad (20)$$

Thus, the lines of magnetic force are circles whose centres lie on the polar axis and whose planes are normal to this axis ; this is otherwise evident.

The r-component of the equation then becomes

$$\frac{\mathrm{H}(r)}{r\sin\theta}\frac{\partial \sin^2\theta}{\partial\theta} = 4\pi(i_0 + i_1\cos\theta),$$

which is true for all values of θ. Hence, we have

$$\left.\begin{aligned} i_0 &= 0, \\ \mathrm{H} &= 2\pi i r \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (21)$$

These equations satisfy the condition that there is no total current leaving the Sun ; the equation $i_0 = 0$ expresses the fact that no steady current could be maintained in a stream emitted uniformly from all over the Sun.

7. To the first order, we find

$$\mathbf{J} = (0, \quad 0, \quad \mathrm{J}\sin\theta), \qquad \Omega = \Omega_0 + \Omega_1\cos\theta,$$

where

$$\mathrm{J} = \frac{e\mathrm{H}}{m_e c} - \frac{1}{r}\left[u_1 + \frac{\partial r v_1}{\partial r}\right] \quad . \quad . \quad . \quad (22)$$

$$\Omega_0 = \tfrac{1}{2}u_0^2 - \frac{e}{m_e}\Phi_0 \quad . \quad . \quad . \quad . \quad (23)$$

$$\Omega_1 = u_0 u_1 - \frac{e}{m_e}\Phi_1 \quad . \quad . \quad . \quad . \quad (24)$$

Inserting these expressions in the equation (8), and retaining only first order terms, the r-component of the equation is

$$\frac{\partial\Omega_0}{\partial r} + \frac{\partial\Omega_1}{\partial r}\cos\theta + v_1\mathrm{J}\sin^2\theta = 0 ;$$

since this is true for all values of θ, we must have

$$\left.\begin{aligned} \frac{\partial\Omega_0}{\partial r} = \frac{\partial\Omega_1}{\partial r} &= 0 \\ v_1\mathrm{J} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (25)$$

each quantity being in reality not zero but of the second order.

The θ-component of equation (8) likewise gives

$$\left.\begin{aligned} u_0\mathrm{J} + \frac{\Omega_1}{r} &= 0 \\ u_1\mathrm{J} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (26)$$

In a similar way we find

$$\mathbf{J}' = (0, \quad 0, \quad \mathrm{J}'\sin\theta), \qquad \Omega' = \Omega_0' + \Omega_1'\cos\theta,$$

where

$$J' = \frac{eH}{m_i c} + \frac{1}{r}\left(u_1' + \frac{\partial r v_1'}{\partial r}\right) \quad . \quad . \quad . \quad (27)$$

$$\left.\begin{aligned} \Omega_0' &= \tfrac{1}{2}u_0'^2 + \frac{e}{m_i}\Phi_0 \\ \Omega_1' &= u_0'u_1' + \frac{e}{m_i}\Phi_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (28)$$

From the equation of motion of the positive ions we deduce similarly:

$$\left.\begin{aligned} \frac{\partial \Omega_0'}{\partial r} &= \frac{I_0}{r^2} \\ \frac{\partial \Omega_1'}{\partial r} &= \frac{I_1}{r^2} \\ u_0'J' - \frac{\Omega_1'}{r} &= 0 \\ u_1'J' = v_1'J' &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (29)$$

each being accurate to the first order only.

The quantities $u_1 J$, $v_1 J$ and the like for the positive ions are thus second order quantities, and hence both J, J′ are, at the most, first order quantities.

8. Considering, finally, the equation of continuity (3), in the present case it can be written

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 n U) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(n V \sin\theta) = 0.$$

Substituting the values of $n\mathbf{V}$ from (11) it becomes, in the case of the electrons,

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \nu_0 u_0) + \frac{\cos\theta}{r^2}\frac{\partial}{\partial r} r^2(\nu_0 u_1 + \nu_1 u_0) + \frac{v_0 \nu_1}{r \sin\theta}\frac{\partial \sin^2\theta}{\partial \theta} = 0.$$

We therefore have

$$\left.\begin{aligned} \frac{1}{r^2}\frac{\partial}{\partial r^2} r^2 \nu_0 u_0 &= 0 \\ \frac{1}{r^2}\frac{\partial}{\partial r} r^2(\nu_0 u_1 + \nu_1 u_0) + \frac{2\nu_0 v_1}{r} &= 0 \end{aligned}\right\}, \quad . \quad . \quad (30)$$

and two similar relations hold for the positive ions.

The first of these equations can be integrated at once to give

$$\nu_0 u_0 = \frac{\text{const}}{r^2} = \frac{C}{r^2}, \text{ say,}$$

and likewise

$$\nu_0' u_0' = \frac{\text{const}}{r^2} = \frac{C'}{r^2}.$$

To satisfy the relation $i_0 = 0$ in (21) we must have $C = C'$.

9. The independent equations to be satisfied are the following :—

$$\left.\begin{aligned} \nu_0 u_0 = \nu_0' u_0' &= \frac{C}{r^2} \\ \frac{1}{r^2}\frac{\partial}{\partial r} r^2(\nu_0 u_1 + \nu_1 u_0) + \frac{2\nu_0 v_1}{r} &= 0 \\ \frac{1}{r^2}\frac{\partial}{\partial r} r^2(\nu_0' u_1' + \nu_1' u_0') + \frac{2\nu_0' v_1'}{r} &= 0 \end{aligned}\right\} \quad . \quad . \quad (31)$$

$$\frac{2\pi e}{c}(\nu_0' u_1' + \nu_1' u_0' - \nu_0 u_1 - \nu_1 u_0) = \frac{H}{r} \quad . \quad . \quad (32)$$

$$\frac{\partial \Omega_0}{\partial r} = \frac{\partial \Omega_1}{\partial r} = 0 \quad . \quad . \quad . \quad . \quad (33)$$

$$u_0 J + \frac{\Omega_1}{r} = 0 \quad . \quad . \quad . \quad . \quad (34)$$

$$\frac{\partial \Omega_0'}{\partial r} - \frac{I_0}{r^2} = \frac{\partial \Omega_1'}{\partial r} - \frac{I_1}{r^2} = 0 \quad . \quad . \quad . \quad (35)$$

$$u_0' J' - \frac{\Omega_1'}{r} = 0 \quad . \quad . \quad . \quad . \quad (36)$$

altogether eleven equations, each being accurate to the first order of small quantities. The independent variables are ν_0, ν_1, u_0, u_1, v_1, the five like quantities for the positive ions, and the magnetic field H—also eleven quantities.

From (33) and (35) we have the following energy integrals :—

$$\left.\begin{aligned} \Omega_0 &\equiv \tfrac{1}{2}u_0^2 - \frac{e}{m_e}\Phi_0 = \text{const} \\ \Omega_0' &\equiv \tfrac{1}{2}u_0'^2 + \frac{e}{m_i}\Phi_0 = \text{const} - \frac{I_0}{r} \end{aligned}\right\} \quad . \quad . \quad (37)$$

the integration of these equations can be completed by using the first equation of continuity in (31). A solution is readily deduced for the case $\rho_0 = 0$; we then have

$$u_0 = u_0',$$

$$\Phi_0 = \Phi_a\left(\frac{a}{r}\right),$$

where

$$e\Phi_a\left(\frac{1}{m_i} + \frac{1}{m_e}\right) = -\frac{I_0}{a} \quad . \quad . \quad . \quad . \quad (38)$$

This relation determines the potential of the Sun necessary to emit the electrons with the positive ions.

The boundary of the stream does not, in reality, extend to infinity, but is a large spherical surface : the condition $\rho_0 = 0$ satisfies the

boundary condition of the problem if we suppose this surface to be uniformly charged with positive charges of density

$$\sigma = -\Phi_a \frac{a}{4\pi R^2} \quad . \quad . \quad . \quad . \quad (39)$$

where R is the radius of the surface. The solution (37) (which is the solution of the problem if $I_1 \equiv 0$) may be written :

$$u_0^2 = u_0'^2 = 2\left(\Omega_0 + \frac{e}{m_e}\Phi_a \frac{a}{r}\right) \quad . \quad . \quad . \quad (40)$$

if we neglect the ratio $\frac{m_e}{m_i}$. This is identical with Professor Milne's formula * if we set

$$\left.\begin{aligned} 2\Omega_0 &= v_\infty^2 \\ \frac{e}{m_e}\frac{\Phi_a}{\Omega_0} &= -\frac{8}{7} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (41)$$

Inserting numerical values, taking v_∞ to be $1{\cdot}6 \,.\, 10^8$ cms./sec., we find

$$\Phi_a = -2{\cdot}7 \,.\, 10^{-2} \text{ e.s.u.} = -8{\cdot}1 \text{ volts.}$$

The electrostatic force at the Sun's surface would thus be only $-1{\cdot}2 \,.\, 10^{-10}$ volts/cm., but yet this is sufficient to enable the electrons to escape with practically the same speed as the positive ions. This numerical example does not hold right up to the Sun's surface, since Professor Milne's formula (8) only holds beyond a distance equal to $\frac{1}{3}a$; it can easily be modified, however, to suit any actual case.

Integrating the equations

$$\frac{\partial \Omega_1}{\partial r} = \frac{\partial \Omega_1'}{\partial r} - \frac{I_1}{r^2} = 0,$$

we find

$$\Omega_1 = \text{const}, \qquad \Omega_1' = \text{const} - \frac{I_1}{r};$$

dividing the first of these equations by (40) we have

$$\frac{u_1}{u_0} = \frac{1}{2} \frac{\Omega_1 + \frac{e}{m_e}\Phi_1}{\Omega_0 + \frac{e}{m_e}\Phi_0}.$$

From (15), $\Phi_1 \equiv 0$ at $r = a$; since $\Omega_0 > 0$, $\Phi_0 < 0$, we have

$$\Omega_1 \not> 2\Omega_0 \frac{u_1}{u_0},$$

and since u_1/u_0 is small, it follows that $\Omega_1 \not> \Omega_0$.

* *Loc. cit.*, p. 464, equation (8).

10. It is now possible to determine the order of magnitude of the magnetic field; from equations (22) and (26) we find

$$\frac{e\mathrm{H}}{m_e c} + \frac{\Omega_1}{u_0 r} = \frac{1}{r}\left(u_1 + \frac{\partial r v_1}{\partial r}\right).$$

The order of magnitude of the term on the right-hand side of this equation is $(|u_1| + |v_1|)/r$, since $\partial v_1/\partial r$ will be of order v_1/r. Since $\Omega_1 \gg \Omega_0$ $(= {v_\infty}^2/2)$, and since u_1 and v_1 are less than u_0 in magnitude, it is clear from the above equation that $e\mathrm{H}/m_e c$ cannot exceed the greater of ${v_\infty}^2/u_0 r$ and u_0/r. Moreover, as $u_0 < v_\infty$,

$$\frac{e\mathrm{H}}{m_e c} \gg \frac{{v_\infty}^2}{u_0 r} \quad . \quad . \quad . \quad . \quad (42)$$

This quantity is greatest near the Sun's surface since $u_0 r$ is least there; the actual value of u_0 depends upon the conditions of emission, and, according to Professor Milne, u_0 is likely to be of the order of 10^6 cm./sec. Inserting also numerical values for v_∞ $(= 1{\cdot}6 \cdot 10^5$ cm./sec.), a $(= 7{\cdot}10^{10}$ cms.) and $\frac{e}{m_e c}$ $(= 1{\cdot}8 \cdot 10^7$ e.m.u.), we find

$$\mathrm{H} \gg 10^{-8} \text{ Gauss} \quad . \quad . \quad . \quad . \quad (43)$$

a value far too small to have any importance in the theory of terrestrial magnetic storms. The current density necessary to produce this magnetic field is, from (20), of the order of 10^{-19} e.m.u., which corresponds to a differential velocity of the ions and electrons of the order of $\frac{10}{\nu_0}$ cms./sec.; it seems unlikely that ν_0 can be very small near the Sun, so that this differential velocity is insignificant in comparison with the individual velocities of the charges. For example, the error introduced by assuming $u_1 = u_1'$ will be 1 in 10^7 if we take $\nu_0 = 1$ ion/c.c.

The further energy integrals, as we have seen, are

$$u_1 u_0 = \Omega_1 + \frac{e}{m_e}\Phi_1,$$

$$u_1' u_0 = \text{const} - \frac{\mathrm{I}_1}{r} - \frac{e}{m_i}\Phi_1,$$

where Ω_1 is a constant; writing $u_1 = u_1'$ in these equations, as is approximately true, we find that

$$e\Phi_1\left(\frac{1}{m_i} + \frac{1}{m_e}\right) = \mathrm{I}_1\left(\frac{1}{a} - \frac{1}{r}\right).$$

The error introduced in Φ, by ignoring the small difference of velocity $u_1' - u_1$, is not greater than 1 in 10^6 if, as before, we take $\nu_0 = 1$.

It is easily verified from (15) that the volume-charge distribution necessary to produce this potential is

$$\rho_1 = \rho\left(1 - \frac{a}{r}\right)\left(\frac{a}{r}\right)^2,$$

where, approximately,

$$\rho = \frac{I_1 m_e}{2\pi e a^3} \text{ e.s.u.}$$

Using the values of Φ_a deduced above in § 9, we find

$$\rho = -\frac{1}{2\pi}\frac{e\Phi_a}{a^2}\cdot\frac{I_1}{I_0} \text{ e.s.u.};$$

inserting numerical values for the various constants, this reduces to

$$\rho = -10^{-22}\cdot\frac{I_1}{I_0} \text{ e.s.u.} < -10^{-22} \text{ e.s.u.},$$

since $I_1 \not> I_0$; the excess positive charge needed to produce this distribution would be, at the most, 1 in 10^{12}; except for extremely rare streams, this is quite insignificant. The greatest contribution to the magnetic field arising from the motion of this charge with the streaming velocity $u_0 (= 1{\cdot}6 \cdot 10^8$ cms./sec.) is, from (32), about $2{\cdot}10^{-14}$ Gauss, which is considerably smaller than the value given in (43). It seems unnecessary to consider the solution of the present problem in greater detail.

We return to consider the more general case of a stream of ions emitted from a restricted region on the Sun's surface. The stream in this case will be laterally limited so that any tendency which the oppositely charged corpuscles may have to separate radially will be counteracted by the electric field produced by the displacement of the charges; as will be seen from the following simple example, it seems unlikely that the transverse equilibrium of the charges, necessary to keep together the positive and negative parts of the stream, can be attained if any current flows within the stream.

Consider an infinitely long cylindrical stream of positive ions and electrons moving together in a direction parallel to the axis of the cylinder, but not necessarily with the same speed. The lines of force of the magnetic field produced will be concentric circles lying in sections normal to the axis, and the oppositely charged corpuscles will separate radially until the electric field, E say, balances the transverse magnetic force on the charges. The condition for this balancing is

$$E = \frac{1}{c}VH,$$

where V is the speed of any charge and H the magnetic field. Since this relation is independent of the particular charge considered, the velocities of the positive ions and electrons must be the same; consequently no current flows in the stream.

During the emission of the stream, the Sun will have a resultant negative charge, necessary to eject the electrons with the positive ions; this charge will spread itself all over the Sun's surface, but mainly over the region around the emitting area. Electrons will thus continually escape from the Sun by electrostatic repulsion; since no total current leaves the Sun, it is necessary that some excess positive charges should

flow down the stream. As we have seen, it seems unlikely that a current will thereby be maintained in the stream; it seems more probable that the latter will carry a resultant positive charge. However, it has been shown * that this would almost immediately diffuse to the surface, leaving the interior of the stream electrically neutral to a high degree of approximation.

These considerations would, therefore, seem to indicate that the solar streams could not, by themselves, exert any appreciable magnetic fields.

On Recombination in Ionized Streams of Corpuscles from the Sun.

By V. C. A. Ferraro.

(*Communicated by Professor Chapman.*)

1. The object of this note is to consider what amount of recombination of ions into neutral atoms is to be expected in an ionized stream of corpuscles during its passage from the Sun to the Earth. This question was examined by Prof. F. A. Lindemann in his paper on magnetic storms; † it is here reconsidered on the basis of Kramers's theory of emission and absorption,‡ taking account also of the geometrical broadening of the stream.

We shall treat the case of Ca^+ ions in detail, but the result is likely to hold good in general.

2. Captures of electrons by a bare nucleus may occur by spontaneous emissions (when the electron is trapped by the bare nucleus) or by induced emission (when the capture is stimulated by the incident radiation). The number of captures per unit time due to spontaneous emission is equal to bn, where b is Einstein's coefficient and n the number of ions per unit volume; the number of captures per unit time per unit volume due to induced emission is $an\mathrm{I}(\nu)$, where ν is the frequency corresponding to ionization and a the corresponding Einstein coefficient.

The ratio of these quantities is $\frac{a}{b}\mathrm{I}(\nu)$; neglecting weight factors (usually of order unity) and supposing the Sun to radiate as a black body at a temperature T of 6000°, we find this ratio to be

$$\frac{1}{e^{h\nu/k\mathrm{T}}-1}.$$

The first ionization potential of calcium is 6·07 volts, corresponding roughly to a temperature of 40,000°. Thus $e^{h\nu/k\mathrm{T}}$ is nearly $e^{20/3}$ and therefore large in comparison with unity; the number of induced emissions is small and may be neglected.

According to Kramers, the probability of capture of a free electron by the different energy levels of a positive nucleus (accompanied by the

* S. Chapman and V. C. A. Ferraro, *M.N.R.A.S.*, **89**, 470, 1929.
† F. A. Lindemann, *Phil. Mag.*, **38**, 669, 1919.
‡ H. A. Kramers, *ibid.*, **46**, 836, 1923.

spontaneous emission of a quantum of radiation $h\nu$) is equal to the probability that the free electron strikes the surface of a sphere, of cross-sectional area β, surrounding the atom : the value of β obtained by Kramers for capture by the nth energy level of the nucleus is

$$\beta = \frac{128 \cdot \pi^4 Z^4 e^{10}}{3\sqrt{3} mc^5 h^7} \cdot \frac{1}{v^2 n^3 \nu} \cdot g',$$

where $-e$ denotes the electronic charge, Ze the charge on the positive nucleus, v the velocity of the electron, m the mass of the electron, c the velocity of light, h Planck's constant, and g' a numerical factor of order unity. The frequency ν is given by the Einstein relation

$$h\nu = W_n + \tfrac{1}{2}mv^2,$$

where $W_n \left(= \frac{2\pi^2 Z^2 e^4 m}{h^2 n^2}\right)$ is the ionization energy of the nth level of the atom ; with sufficient exactness we may take v to be equal to the thermal velocity of the electrons at the temperature of the gas.

Inserting numerical values for the various constants, taking $Z = 1$ and the temperature to be 6000°, we find

$$\beta = \frac{1{\cdot}2 \cdot 10^{-19}}{n + 4{\cdot}5 \cdot 10^{-2} \cdot n^3} g' ;$$

the length of the mean free path, corresponding to a probable capture of a free electron by the nth level of the nucleus, is accordingly

$$\lambda_n = \frac{1}{N\beta} = \frac{8{\cdot}3 \cdot 10^{18}}{Ng'} (n + 4{\cdot}5 \cdot 10^{-2} \cdot n^3) \text{ cms.},$$

where N denotes the number density of the free electrons in the gas. The formula shows that we must expect captures in the higher levels of the atom to be considerably rarer than in the lower ones ; we can, therefore, define the mean free path of electron capture, λ_c, by putting $n = 1$ in the above expression for λ_n and omitting also the factor $\frac{1}{g'}$. We find

$$\lambda_c = \frac{8{\cdot}3 \cdot 10^{18}}{N} \text{ cms.} ;$$

this is of the same order of magnitude as the value given by Professor Lindemann.

Each termination of this free path results in the probable capture of an electron ; the number of recombinations per sec. per c.c. is therefore $\frac{Nv}{2\lambda_c}$ or, substituting for λ_c, $3 \cdot 10^{-12} N^2$ collisions per sec. per c.c. ; the coefficient of recombination α is accordingly

$$\alpha = 3 \cdot 10^{-12} \quad . \quad . \quad . \quad . \quad (1)$$

The expression deduced above may be used, in general, to determine the rate at which singly ionized atoms recombine, through collisions with electrons, in a gas at a uniform temperature of 6000°.

3. To determine completely the rate at which ions and electrons recombine we should also inquire what the rate of ionization of the atom by the incident radiation is likely to be; for our purpose, however, this point seems to be of no immediate importance, and we shall calculate the amount of recombination to be expected at the arrival at the Earth of an ionized stream of corpuscles from the Sun, ignoring altogether the solar ionization.

Consider a small volume element of the stream, δV, which, at all times, contains the same number of particles. Let N be the number density of the fraction of ionized atoms in this element at any time, and let the suffix o refer to the initial values of N and δV; in our case these will be the values at the Sun's surface, $r = a$ say.

The respective values of the number density of the ions and of the volume of the element at two successive instants will be

$$N, \delta V; \quad N + dN, \delta V + d(\delta V).$$

If no recombinations took place during this small interval of time, the total number of ions in the element would be the same at each instant; the difference

$$N \,.\, \delta V - (N + dN)(\delta V + d(\delta V))$$

must, therefore, be equal to the number of ions recombined in the element during the small interval of time, dt say. This is also equal to $\alpha N^2 \,.\, \delta V \,.\, dt$; hence, equating these two expressions, and proceeding to the limit as $dt \to 0$, the equation giving the rate of change of N due to both recombination and dilatation of the volume element is

$$\frac{dN}{dt} = -\alpha N^2 - N\frac{1}{\delta V} \cdot \frac{d(\delta V)}{dt} \quad . \quad . \quad . \quad (2)$$

If δV is constant the equation reduces to the ordinary form, whilst if $\alpha = 0$, in which case there are no recombinations, the equation can be integrated at once to the ordinary equation of continuity.

4. Professor Chapman has shown * that owing to geometrical broadening of the stream, δV will increase proportionately to the square of the distance from the centre of the Sun, r say. Thus, we have

$$\delta V = \delta V_0 \left(\frac{r}{a}\right)^2.$$

Beyond a few solar diameters the velocity, v, of the corpuscles will be approximately constant, and we may write, with sufficient exactness,

$$r = a + vt.$$

Supposing that the atoms in the element are all initially ionized, the equation (2) can be integrated simply. The solution may be conveniently written in the form

$$\frac{N_0 \,.\, \delta V_0}{N \,.\, \delta V} - 1 = \alpha \frac{a}{v} N_0 \left(1 - \frac{a}{r}\right).$$

* *M.N.R.A.S.*, **89**, 456, 463, 1929.

The left-hand side of this equation is a measure of the fraction of recombined atoms in the volume element. It continually increases as the element recedes from the Sun, but tends to a definite limit, instead of to infinity, at infinite distance. This is because the decrease of N due to geometrical broadening will, after a comparatively short time, become more important than the diminution of N due to recombinations; in other words, the latter become less and less effective as the volume element increases. This can be seen by writing equation (2) in the equivalent form

$$\frac{d}{dt}\left(\frac{1}{N \cdot \delta V}\right) = \frac{\alpha}{\delta V},$$

α being constant.

In our case $\alpha = 3 \cdot 10^{-12}$, and inserting numerical values for a ($= 7 \cdot 10^{10}$ cms.) and v ($= 1{\cdot}6 \cdot 10^{8}$ cms./sec.) we find at the Earth's orbit

$$\frac{N_0 \cdot \delta V_0}{N \cdot \delta V} - 1 \sim 1{\cdot}4 \cdot 10^{-9} N_0.$$

Thus, provided that N_0 does not exceed 10^9 ions/c.c. at emission, recombinations of ions into neutral atoms can be neglected. In denser streams, though the number of recombinations would be very much greater, the number density of the ions left in the stream element could not be less than

$$\frac{\delta V_0}{\delta V} \cdot \frac{1}{\frac{1}{N_0} + 1{\cdot}4 \cdot 10^{-9}};$$

at the Earth's orbit, $\delta V_0/\delta V = \left(\frac{a}{r}\right)^2 = \frac{1}{46{,}000}$, and as we suppose $N_0 >> 10^9$, this expression is not less than about 15,000 ions/c.c. This point is, however, only of theoretical interest, as it seems unlikely that streams of corpuscles as dense as this can be emitted from the Sun. The above must be a considerable underestimate in the case of very dense streams since we have neglected the solar ionization.*

Two New Bright-hydrogen-line B5-type Stars. By William J. S. Lockyer, M.A., Ph.D., Major (late R.A.F.).

(1) H.D.C. 138749. θ CrB. 1900 R.A. $15^h\ 28^m{\cdot}9$. Dec. $+ 31^\circ\ 42'$. Mag. 4·17. Sp. Type B5.

In September 1923 I began a special investigation of re-photographing systematically all the available bright-hydrogen-line stars of B-type which Paul W. Merrill had observed during the years 1911 and

* This and the preceding paper form part of a thesis for which the degree of Ph.D. was awarded by the University of London.

1912 at the Lick Observatory,* in order to record any changes that might have taken place in their spectra during the interval. This investigation is still in progress.

During the course of this work one was naturally led to examine the spectra of many other stars of the same spectral type to detect, if possible, bright lines which may have escaped the notice of other observers.

Prior to the above-mentioned research the 9-inch prismatic camera of the Norman Lockyer Observatory had been employed in securing the spectra of stars of various spectral types within the range of the optical capacity of the instrument, and 350 negatives had been secured for discussion. Incidentally a large number of B-type stars were included in this total, and in carefully examining these I found in 1928 that the spectrum of the star θ Coronæ Borealis exhibited bright hydrogen lines.

Unfortunately the photograph, which was taken on 1922 June 12, is not very dense, and consequently Hβ is too under-exposed to exhibit bright lines. On the other hand Hγ displays what appears to be a single bright line on a wide absorption band, and both Hδ and Hϵ show double bright lines on wide absorption bands with the violet a little stronger than the red components; a bright double line at Hζ is doubtfully present. The other few lines present in the spectrum are the absorption lines of helium, which are rather weak and very nebulous, and the lines of ionised magnesium (λ 4481) and Calcium (K) which are very weak. There are no traces of the sharp enhanced metallic lines which are so conspicuous in the Bo-type star ϕ Persei.

This star was placed on my observing list in 1928, and during that year four photographs were secured. None of these was very dense, but while Hγ, Hδ, and Hϵ exhibited bright lines their duplicity was doubtful.

The star was not photographed in 1929, but during the current year 21 photographs were secured between March and October inclusive. Several of these show that the hydrogen lines have bright components, but in all cases they are faint. The best photograph, taken on May 18, shows that each of the lines Hβ, Hγ, Hδ, Hϵ, and Hζ exhibits a faint double bright line symmetrically placed on each wide absorption band, and that the bright components are of about equal intensity in each case or perhaps the violet is a shade stronger than the red component. As the hydrogen bands become broader towards the violet end of the spectrum, the double bright lines become gradually more conspicuous in that region.

In order to find out whether any other observers had recorded bright lines in the spectrum of this comparatively bright star, references were made to all the available sources of information.

E. C. Pickering, in his *Revised Harvard Photometry*,† which was published in the year 1908, makes no mention of bright lines in his "Remarks," as is usual when bright lines have been seen.

* *L.O.B.*, **7**, 162–179, 1913. † *H.C.O. Ann.*, **50**, 133, 1908.

Paul W. Merrill, in his dissertation on "Class B Stars whose Spectra contain Bright Hydrogen Lines," * evidently recorded no bright lines, because he refers to the spectrum he secured in 1911 May 21·74 as "Strong Absorption Hα, Hβ. Faint λ 4922 Absorption."

J. H. Moore makes no mention of this star in his "Third Catalogue of Spectroscopic Binaries," † and Paul W. Merrill, Milton L. Humason, and Cora G. Burwell,‡ who made a thorough investigation entitled "Discovery and Observations of Stars of Class Be," have not entered this star on their very extensive list.

Edwin B. Frost, Storrs B. Barrett, and Otto Struve, in their investigation on the "Radial Velocities of 368 Helium Stars," § measured the lines in the spectrum for radial velocity in 1904, 1917, and 1918, but, apart from saying that the lines were few and very poor, and the measures very uncertain, they made no reference to any bright hydrogen lines.

W. W. Campbell and J. H. Moore in their volume on the "Radial Velocities of Stars" || also did not record any bright lines, for they refer to the spectrum as containing "only very broad hydrogen and helium lines."

Lastly, Ralph H. Curtiss, who made a study of the spectra of Be-type stars, makes no mention in the publications of the Detroit Observatory of having observed this star at all.

The above references indicate therefore that the discovery of bright lines in the spectrum of this star here recorded is the first mention of their presence.

Of course it cannot be definitely stated whether these bright hydrogen components have recently made their appearance or whether they have been there all the time but not been noticed. Since they are recorded on a photograph I secured in 1922, were still visible in 1928, and are there now (October 1930), it may be assumed that they have been at any rate continuously present since 1922.

Probably one reason why they have not been detected before is that, being rather faint, they require not only a dense negative to bring them out but also, and perhaps more especially, a night of very good definition. On several occasions I have secured photographs which were sufficiently dense to show the lines, but they were found to be absent: this I considered to be due more to the lack of good definition than to their actual non-existence.

Another very possible reason is that the large dispersion of the spectra taken with the 9-inch prismatic camera with a prism of 45° angle (46·0 mm. between Hβ and Hδ) is very suitable for recording such monochromatic images, and that that used by other observers has not been so great.

When much less dispersion is used, such as that with the 12-inch prismatic camera with a prism of 20° angle of the Norman Lockyer Observatory, the presence of such bright lines cannot be detected at all unless the lines are fairly strong.

* *L.O.B.*, **7**, 171, 1913. † *L.O.B.*, **7**, 141, 1924. ‡ *Ap. J.*, **61**, 231, 1924.
§ *Ap. J.*, **64**, 15, 1926. || *L.O.P.*, **16**, 226, 1928.

The spectral nomenclature of this star, according to the ruling made by the International Astronomical Union, will now have to be written as B5*neζ*.

Finally it is interesting to note that, while it is now known that stars of spectral classes Boe to B5e tend strongly to cluster towards the Galactic Equator,* this star forms one of the few exceptions to this law, for it is situated about 50 degrees away from the nearest part of the Milky Way.

(2) H.D.C. 224544. And. 1900 R.A. $23^h\ 53^m{\cdot}7$. Dec. + 31° 48′. Mag. 6·36. Sp. Type B5.

In a search for bright-hydrogen-line stars among the B-type stars whose spectra were photographed at the Norman Lockyer Observatory with the 12-inch prismatic camera having a prism of 20° angle (dispersion 21·4 mm. between Hβ and Hδ), it was found quite recently that, on three negatives of this star taken on 1925 December 1 and 3, the Hβ line exhibited a distinct bright line in the centre of the broad absorption band. The dispersion is such that it cannot be said whether this line is single or double. This star is out of range of the 9-inch prismatic camera, so that the duplicity or otherwise of the line cannot be ascertained. The other hydrogen lines, Hγ, Hδ, and Hε, suggest narrow absorption lines on broad absorption bands, and this means that probably each of these narrow lines has a bright line on each side of it, as is customary in this class of spectrum, and that consequently Hβ has a double bright line.

As the above negatives were taken by Mr. D. L. Edwards in the routine programme for the work on spectroscopic parallaxes, I pointed out the bright-line nature of Hβ to him, and he informed me that he had also noticed it when examining the spectra.

In order to obtain, if possible, better negatives than those referred to above, the star has been very recently photographed again on the nights of October 18 and 25 last, and broader and denser negatives were secured with exposures of one hour and forty minutes.

The bright-line nature of Hβ is again clearly demonstrated, and Hδ and Hε exhibit narrow absorption lines on broad absorption bands as before. Hγ behaves in these photographs like Hβ, only the bright line is very faint.

Apart from the hydrogen lines the general spectrum is practically a replica of that of θ Coronæ Borealis, the star previously discussed.

No trace can be found of any mention of Hβ, or even Hα, as having been observed bright before, in spite of the comparatively great intensity at the present time of the bright line at Hβ. Pickering gives no record of any bright lines, neither is there any reference to them in the Henry Draper Catalogue. In all the other references given above in the case of θ Coronæ Borealis no mention is made of this spectrum at all.

The spectral nomenclature of this star will now have to be written as B5*neβ*.

* R. H. Curtiss, *J.R.A.S. Can.*, **20**, 31, 1926.

The position of this star in relation to the Milky Way is about 10 degrees from its boundary. As it is in that part of the constellation of Andromeda which is closely adjacent to that of Pegasus, it looks as if it may form part of the group of stars in the Capricornus-Aquarius-Pisces-Pegasus region, which region lies well away from the Galactic Equator, and the star thus provides an outstanding exception in the distribution of this special type of spectrum.

A New Photo-electric Recording Microphotometer. By J. A. Carroll, M.A., Ph.D., and E. B. Moss, B.Sc. (Plates 1–5.)

(*Communicated by the Director, Solar Physics Observatory, Cambridge University.*)

In pursuing a programme of investigation at the Solar Physics Observatory, Cambridge, on the contours of stellar and solar spectrum lines and in measuring the distribution of radiation in the nebulæ, it was found that no existing microphotometer fulfilled conveniently the special requirements of this work and also that involved in the spectrophotometric determinations made at total eclipses of the Sun.

Accordingly, these special requirements being kept in mind, a new photometer has been designed during the past two years, in collaboration with the Cambridge Instrument Company, which performs all the functions of existing types of instrument and has also greater speed of working, greater sensitivity and increased adaptability and range. The instrument has been designed to meet the following requirements :—

1. Measurement of spectrum lines, both along and at right angles to the direction of dispersion.
2. Measurement of extended objects, (*e.g.* nebulæ and the corona), and plate sensitometry.
3. Stellar photometry.
4. To make measurements at any point of a plate 12 inches by 12 inches, and in any desired direction.
5. To obtain on the record reference marks giving directly the position on the plate measured.
6. To be operable at commonly occurring densities with an effective slit width of ·005 mm., length 0·5 mm.
7. To be useable as a direct reading instrument.

1 requires as a rule the use of a very narrow slit, and consequently, if high densities (*i.e.* greater than 2) are to be measured, great sensitivity. Also, while in general it is desirable to use as long a slit as possible to average out the grain of the image on the photographic plate, in stellar spectrophotometry the length to be used can rarely exceed 0·5 mm. and may be considerably less.

2 and 3 require the use of small apertures of special shapes, usually circular or rectangular.

The instrument in the main is an adaptation of the Dobson-Skinner microphotometer, and Plates 1 and 2 and figs. 1 and 2 show the general design and appearance. The optical train is illustrated diagrammatically in fig. 1.

The filament of the standard lamp L is focussed on the wedges W, between which a V-shaped diaphragm (V) limits the length of filament used—that is, ultimately the effective length of the slit is so fixed. The microscope objective O_1 forms a reduced image of the filament on the

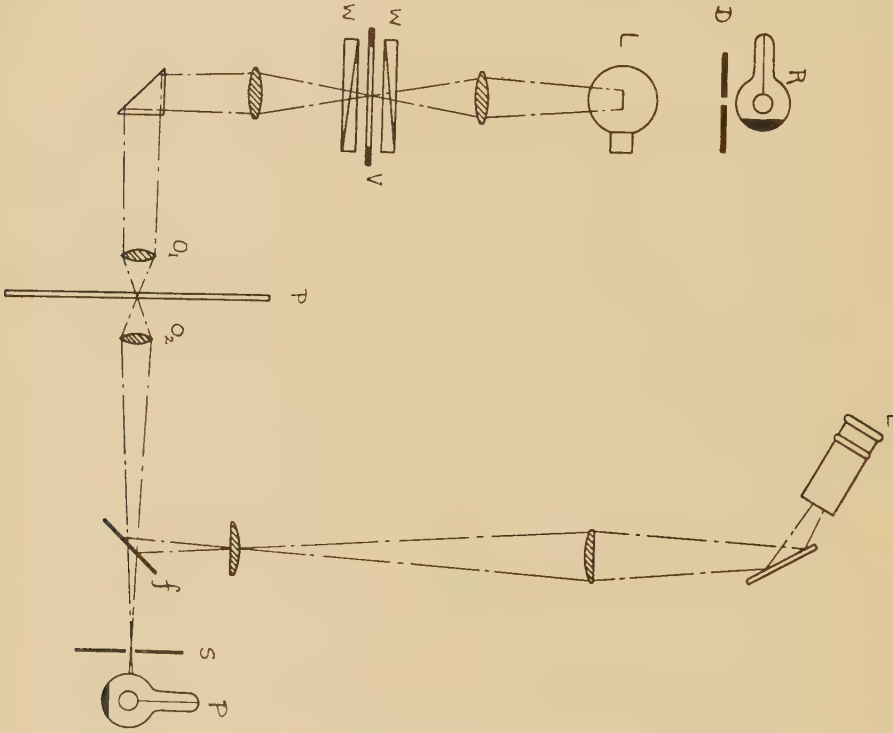
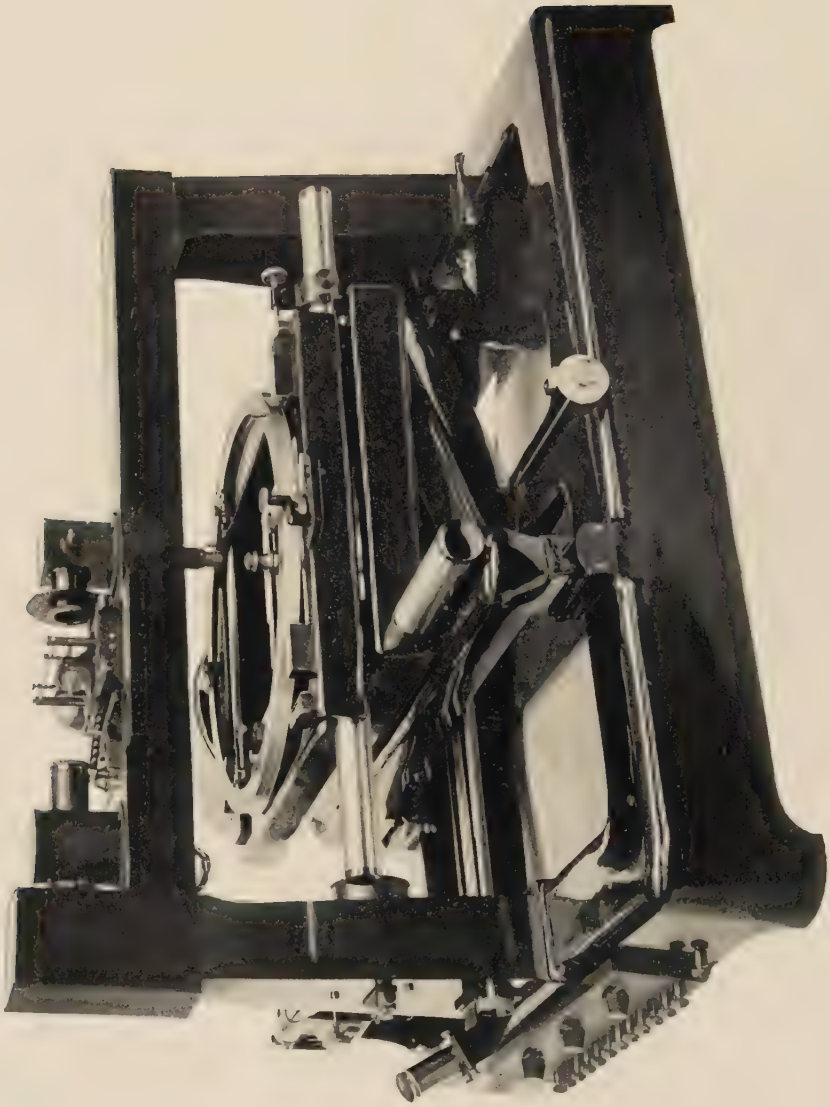
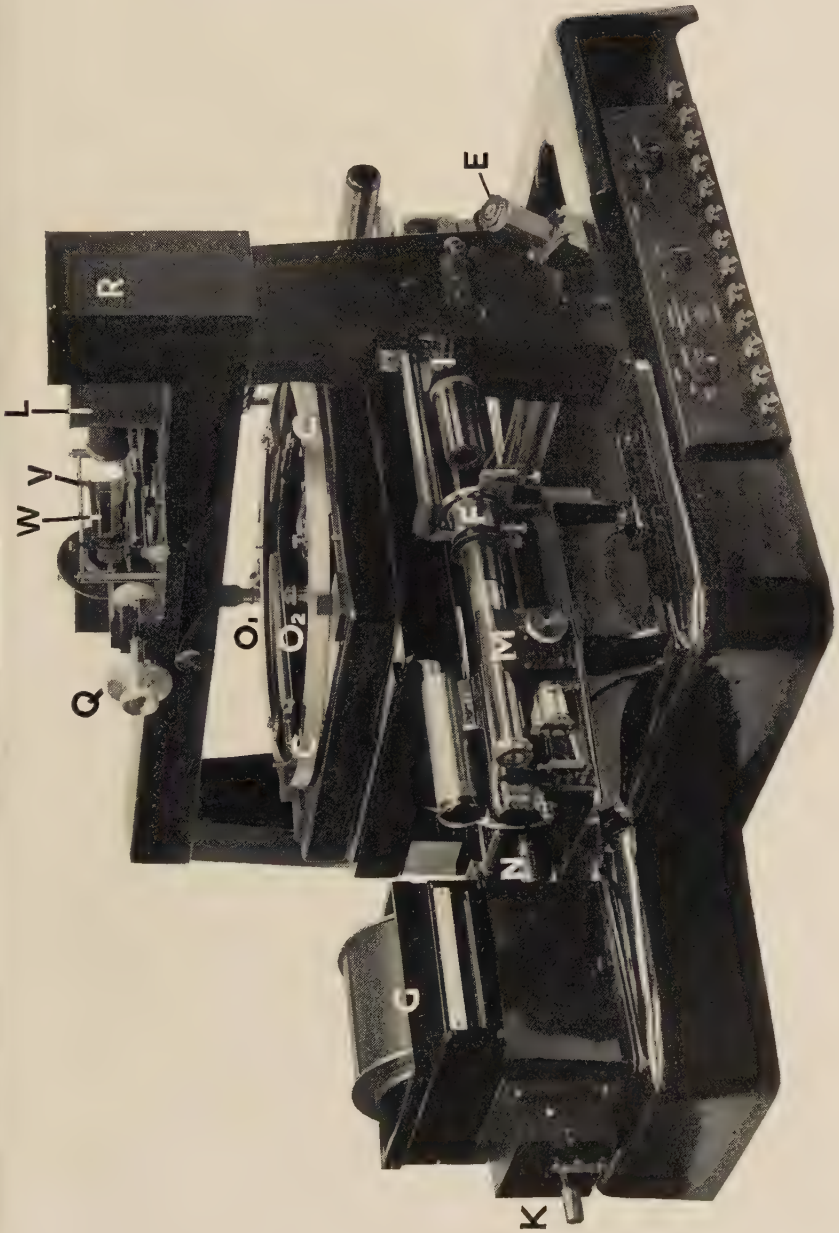


FIG. 1.

plate P, which is to be measured, held in the holder C. An enlarged image of the area of the plate illuminated is projected by the microscope objective O_2 on to the slit S, and a thin collodion film *f* serves to send a part of the light to an eyepiece E for inspection of the part of the plate under measurement, and also for the purposes of adjustment.

The slit S can be adjusted to any desired width (not greater than the width of the magnified image of the filament formed on it), and the area of the plate under measurement at any instant is given by the length of the image of the filament on the plate and the width of the image of the slit that would be formed on the plate by the lens O_2. This latter is arranged to give a magnification of about 12 times, so




R
E
L
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V
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O1
O2
C
F
M
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G
K

that the actual width of the slit for even the finest work is still so great that inevitable inaccuracies in its edges have little or no effect and its width can be taken to be sensibly uniform.

The light from the area under investigation passes through the slit and falls on a photo-electric cell P. This cell is the measuring cell and is an Osram cæsium on silver cell, type CMG 7—a great advantage, since it is sensitive well into the infra-red, and, moreover, still operates with ease when colour filters are placed in the beam of light from the standard lamp L. The cell is connected to a Lindemann electrometer F, with a special needle and pointer to make it suitable for projection, through a second cell R, which serves as a variable high resistance. This cell R is illuminated from an iris diaphragm D at the standard lamp L, and in this way a compensation is obtained for small variations of the standard lamp L, provided that the two cells have cathodes of similar material. It will be appreciated that, apart from the use of the Lindemann electrometer in place of one with a string, this arrangement follows that described by P. P. Koch (*Ann. der Phys.*, **29**, 705, 1912). The standard lamp uses 4 to 6 amperes at 6 volts and is fed from cells capable of giving fifty times this current; they are, moreover, kept well charged and allowed to run a short while before use to avoid the drop in potential that occurs on first using a fully charged accumulator. Under these conditions the potential across the lamp remains very steady, but should a small change occur the illumination of R changes also, and hence its resistance, and in the opposite direction to the change of intensity of illumination. By a suitable choice of the disposable constants good compensation may be obtained over the whole working range of the instrument. The diaphragm D also serves as a simple means of altering the resistance of R and so changing the damping and sensitivity of the instrument.

An independently fed lamp I (fig. 2) illuminates the needle of the electrometer, an image of which is projected by the microscope M on to the recording drum G. The slit and filament F are normally set at right angles to the direction of travel of the plate carrier C. By changing the objectives O_1 and O_2, and by replacing S by suitable apertures, any desired shape or size of slit can be obtained.

To obtain the necessary rigidity, C and its supports must be heavy, and so for ease of construction an arrangement is used in which the plate is horizontal and moves in a horizontal plane. The recording drum can accommodate about 10 cm. of the plate, and accordingly the mechanical drive is restricted to this amount. The centre of the region to be measured can be at any part of a plate 12 inches square, however, and this motion is secured by providing worked parallel bars and plane surfaces on which the plate carrier moves in two directions at right angles. It can be clamped in any position and a fine adjustment is provided by micrometer screw. Scales are provided for rough setting to $\frac{1}{10}$th mm., and a viewing telescope of low power, placed at one side near the wedges W, enables the image of the filament on the plate to be seen and a considerable region of the plate around it. The top table is capable of rotation through 360° and is graduated around its edge;

settings to 0·1° can readily be made. The plate itself is held by moveable slides carrying clamps, and three screws provide a means of securing that the plane of the plate is perpendicular to the optic axis of the projection system O_1, O_2, so that the film is in focus at all points of the plate in any azimuth. In operation the plate is made to move slowly at a uniform speed in a fixed direction and is geared to the recording drum, on which a record of the deflection of the electrometer needle is

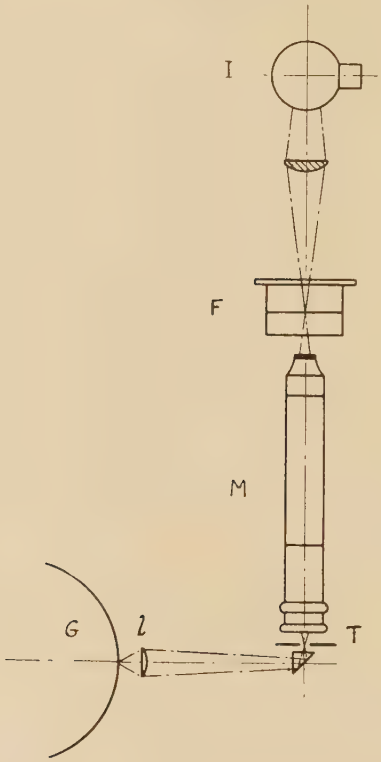


FIG. 2.

traced on bromide paper. This gear ratio may be either 10 to 1, *i.e.* one mm. on the plate is spread out into 1 cm. on the record, or 50 to 1. This is found to be adequate for almost all needs, but of course any desired ratio could be obtained in place of one or other of these.

The Recording System.

The plate carrier is driven by a motor attached at K, coupled through reduction gearing (which is not variable, but the speed of operation may be varied to suit different sensitivities of the apparatus by means of change-over pulleys on the belt-drive coupling to the

motor) to the shaft N, which consists of the driving worm which moves the plate through a nut, and the rollers driving the recording drum. These rollers and the worm shaft are in one piece ; thus any slip is avoided. The drive of the drum is by friction, errors in gear teeth being thus avoided. The arrangement of the drum drive is illustrated in figs. 3 and 4. The drum carries the paper, which is 12 cms. by 50 cms., and is friction driven by the shaft N through the ground rollers *aa* or *bb*. Two magnifications are thus obtainable by displacing the drum sideways so that AA and *aa* bear together, or BB and *bb*. This device is found to

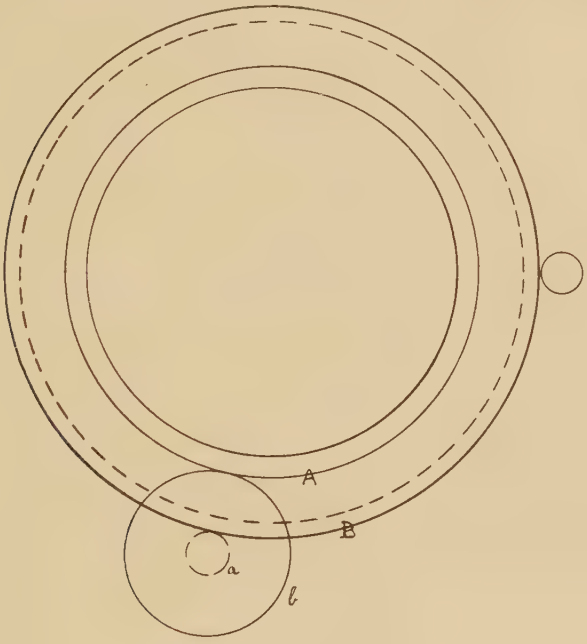


Fig. 3.

work excellently. The drum can also be raised off the driving shaft and held stationary while the plate is moved any desired amount forwards or backwards. A cylindrical lens *l* (fig. 2), with vertical lines engraved on it every mm., focusses the images of the needle on the bromide paper, and there results on development a black ground crossed by a series of white lines, providing a scale for measuring deflection of the electrometer needle and also the trace of the needle itself. Every fifth line is somewhat heavier than the intermediate ones. A specially useful device consists of a shutter T (fig. 2) operated by electrical contacts in the driving gear, which occludes the light from the projecting microscope every mm. of travel of the plate, and so gives on the record itself a white line at right angles to the other system, indicating absolute position on the plate, measured from some suitable zero point

obtained by ruling a fine black line on the plate itself. It is noteworthy that two records taken over the same part of the plate in the *same* direction show that accurate register is maintained between position on the plate and the 1-mm. white lines on the record. This means of indication on the actual record of position on the plate is invaluable, especially for such investigations as those bearing on the lines of equal density in the image of a nebula.

It may be noted here that the paper clip on the drum is enlarged at

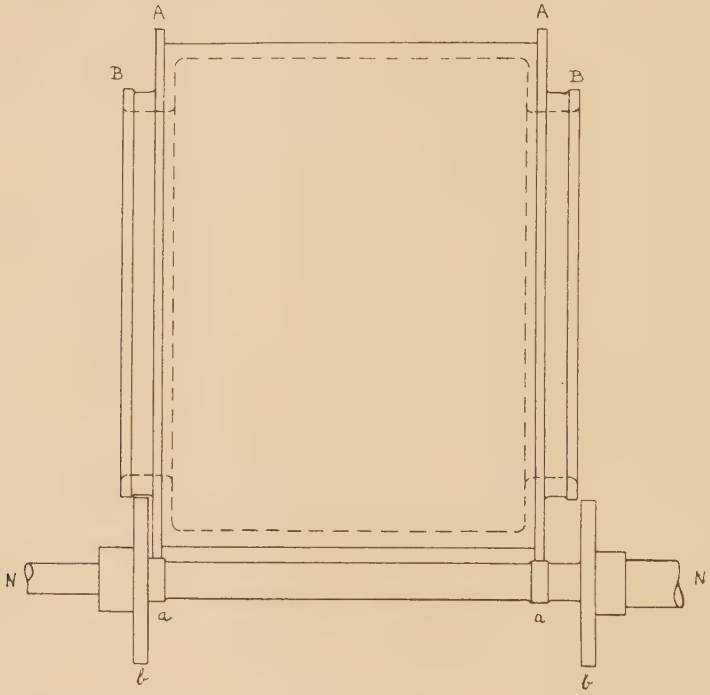


FIG. 4.

one end, so that in the course of rotation it operates the switch mechanism so placed that the driving motor is stopped after one revolution of the drum. The removal of the drum for re-loading would allow the motor to start were not the contact provided with a ratchet mechanism, and the motor is only released by pressing the small knob in front of the camera. Limit stops are also provided to the travel of the nut on the micrometer screw.

The Wedges.

Two equal uniform wedges are used with their gradients in opposite directions so that a patch of uniform density of any amount from zero to the sum of the two wedges can be interposed in the path of the beam from the standard lamp. One of the wedges can be moved by a micro-

Fig. 2.

Fig. 3.

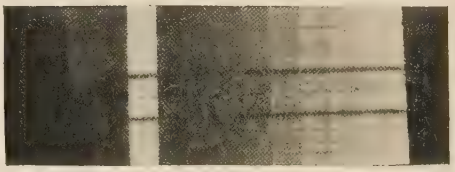


Fig. 1.

meter screw Q, thus enabling the instrument to be used without the recording system for direct measurements and as a null instrument with very high sensitivity. The other can be moved in fixed steps only, and these are arranged to give an increase in density of 0·5 at each step. Alternatively, a third wedge can be added that has a succession of uniform strips increasing in density by 0·5 each time. The last-named is preferable as being not liable to change or disturbance. The purpose of these stepped wedges is to place on the plate, say before and after each run, marks indicating the deflection produced by different known densities. It sometimes happens, when one is working with the highest sensitivities, that a change in density of 0·5 or even less will produce a deflection of the whole extent of the scale, and if the density of the plate varies over the regions required by something of this order then the needle can be brought on to the scale again by moving the wedge and adding a convenient *known* density to the plate thereby. Of course no provision is made for recording this act of the operator, and a note must be made of what has been done.

Operation of the Instrument.

From what has gone before, the *modus operandi* of the instrument will be fairly clear and a brief summary only will be given here, no account of the adjustment of the electrical system being undertaken. This latter is in no way difficult or special as it resembles the adjustments required in using an electrometer for any other purpose. It is merely necessary to see that the required sensitivity is obtained, neither more nor less, and that the speed of operation is not set too fast. The full-scale deflection can be obtained, it should be noted, for as little as a change of 0·05 in density, or for as much as 3·0. The instrument may be run at full speed except for sensitivities of less than 0·5 for full-scale deflection. It is built to travel at 6 mm. of the plate per minute. A high speed is provided for moving the plate about rapidly at three or four times this rate, it is not intended for use in recording, but only to save time in setting the plate.

The period of oscillation of the electrometer needle for the common adjustment is about 1 second, and it is nearly critically damped. For very narrow lines it will be necessary to run at a somewhat slower speed, but this is not often necessary. Like any photometer, the instrument has great sensitivity at low densities, but this depends also on the strength of the illumination.

The successive steps in operation, if the sensitivity be known to be suitable, are then:—

1. Put in plate and level up and align it (in azimuth).
2. Find zero mark and arrange to traverse it.
3. Operate the controls on the control panel.
4. Move wedge if necessary during the run.
5. Put on calibrating steps by moving wedges with plate removed or over clear glass.
6. Develop.

Illustrative Examples.

1. *Contours of Solar Fraunhofer Lines.*—In Plate 3, fig. 1 shows a reproduction of a negative taken with the McClean solar spectrograph of the D lines. A stepped wedge placed over the slit gives a sequence of strips of spectrum at right angles to the direction of dispersion, with a known ratio of intensity during the actual exposure. This known ratio is obtained by an independent calibration of the wedge, for which purpose the recording microphotometer may be used as an absolute photometer by the use of suitable colour screens and a standard absorbing screen.

The record obtained by running one of these strips through the microphotometer is shown in fig. 2. The right-hand portion shows the trace of the microphotometer needle over the lines. The left-hand portion * shows the result of moving the stepped wedge W of the microphotometer through amounts adding a density of 0·5 at each step. Hence the density of the image at all wave-lengths can be read off. Accurate measurement of the wave-length at a point is made easy by the vertical white lines equivalent to 1 mm. on the actual plate. It would suffice to take one more record run in a direction perpendicular to that of dispersion in the neighbourhood of the D lines in order to obtain the relation between the density and the intensity of illumination of the plate for this exposure. It is, however, better and more accurate to proceed as follows. Every one of the strips of spectrum is run through in the same way, the sensitivity of the instrument being adjusted to give nearly full-scale deflection for the change in density of the D lines in the particular strip examined. The density is known then at all points for each intensity, and a curve may be drawn relating blackening and intensity for each wave-length. Over so small a range of wave-length these curves will be sensibly the same, and by drawing several a very good determination may be made of the true curve, and, moreover, it may be extended at both ends beyond the range of the wedge on the spectroscope by using the variation over the lines. Corrections must be made for plate fog and for scattered light. This can be done by measuring over an unexposed strip adjoining the spectrum, and precision is reached by extrapolation from the curve found as above. It does not follow that the same increase in density is produced by the addition of a given amount of illumination at all values of the illumination, and by the complete system of measurement described this can be allowed for, since the intensity curves of the lines themselves must be the same in all the strips. Finally, a curve can be drawn representing the mean of all the determinations from the individual strips, in the course of which errors due to variation of the plate will be smoothed out. The result will be an accurate determination of the distribution of the light incident on the plate, neglecting certain " photographic " corrections such as scatter in the plate itself, which must be determined separately. It does not follow that this is the same as the distribution of light in the spectrum of the light incident on the spectroscope slit.

* Fig. 3 of Plate 3 shows the right-hand portion of fig. 2 taken with a higher sensitivity of the microphotometer.

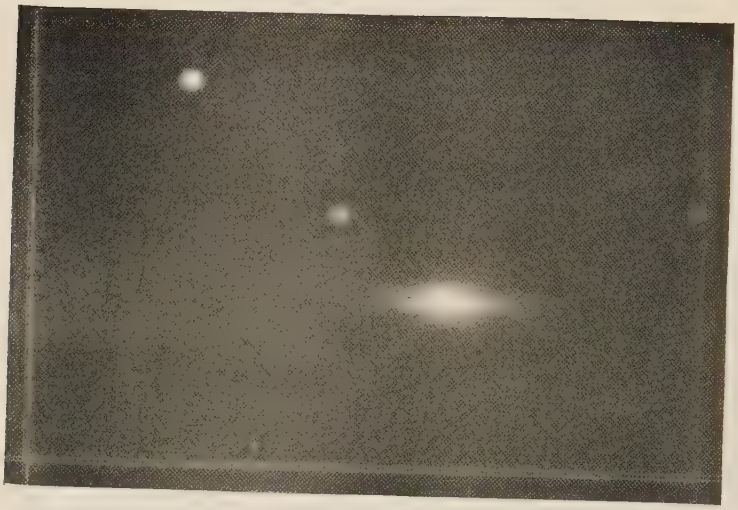


Fig. 1.

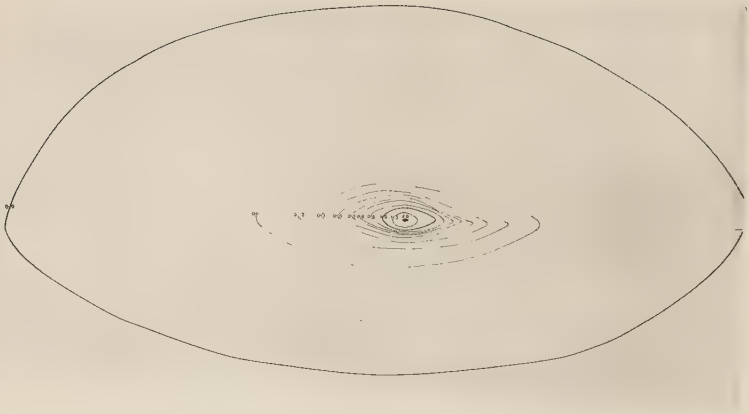
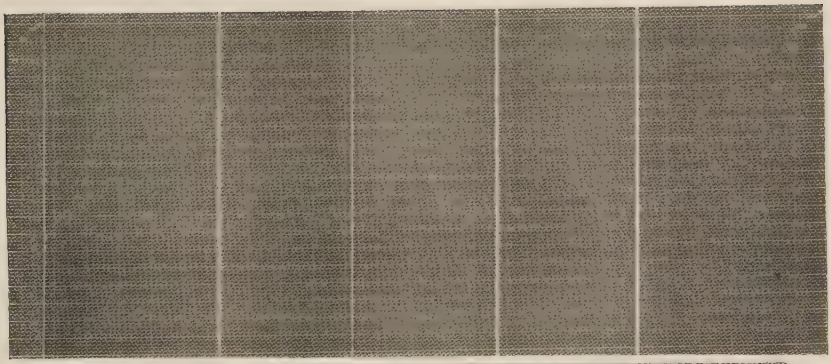


Fig. 2.

. 3.



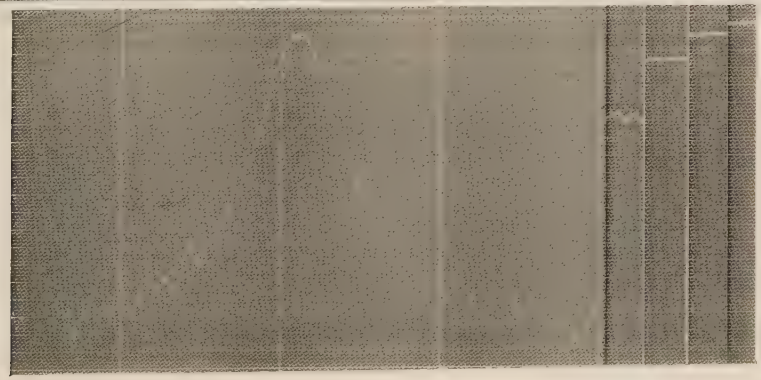


Fig. 4.

Various effects, such as that due to the finite resolving power of the spectroscope, must be measured and allowed for, but the microphotometer has completed its task in measuring the plate given it; no further corrections are required in the use of it.

2. *Isophotic Lines in a Nebula.*—The procedure, in so far as determination of density is concerned, is the same as described for a spectrum in 1 above. The plate is first oriented by means of the rotating carrier so that the runs are along the major or minor axis of the nebula or at some known angle to the long axis. By making short runs across the axis and noting the position of maximum deflection, using not the recording motion but the fine adjustment of the carriage at right angles to the direction of travel while recording, an accurate determination of the position of the axis may be made. By means of the fine adjustment to the carriage, runs may be taken at selected intervals on either side of the axis, *i.e.* sections across the nebula from end to end, say, every 0·5 mm. from the median line. To secure accuracy of fit of the sections it is well to make a series of runs in the perpendicular direction also. This will assist in smoothing the contour lines of equal density. Two lines ruled on the plate at right angles near the nebula should be included in the record as reference marks for measurement, and the white lines, giving every mm. of plate travel, suffice to ensure accurate "register" of the records. If numerical values of the "isophotic" lines are to be found, *i.e.* lines of equal brightness, the plate must have on it an image impressed through an optical wedge of some kind to enable the relation between intensity and density to be found for the particular plate and exposure.

The process is illustrated in Plates 4 and 5. It will be noticed that the extent of the nebula as revealed by the microphotometer is much greater than can be seen by eye on the plate.

3. *On the Waviness of the "Background Line" and Plate Grain.*—If a trace be obtained with no plate in the instrument it will be seen that it is quite steady, and one edge or the other may be taken as the recording line and read to one-tenth of a division easily. This is the case whatever the sensitivity of the instrument. If the sensitivity of the instrument be such that the full-scale deflection is obtained for a density change of 2·0 to 2·5, and a developed but unexposed or very slightly fogged plate be inserted, the record is found to be wavy, the waviness being least when a density of, say, 1·5 to 2·0 has been added by the movable wedges and greatest for the film solo. This waviness is due to the well-known grainy structure of the photographic image, and its presence is evidence of the sensitivity of the instrument. For a given area of photometer slit the amplitude of this variation represents the mean fluctuation of density over that area about the constant value it would have if the plate were perfectly uniform and of infinitely fine grain. It can be diminished by increasing the area of the slit or by using finer grained plates. The matter is in any event one of compromise, since the area of slit is limited, first, by the width of the spectrum perpendicular to the direction of dispersion; and, secondly, by the width of slit necessary to secure adequate resolving

power over the line measured, *i.e.* it must not be so wide that the density gradient changes appreciably over its width. Fine-grained plates are slow, and in many spheres of work slowness cannot be tolerated. For a given plate and a given size of slit the waviness depends on the density of the photometer wedge plus the plate. The reason for this is that the fluctuations represent a given nearly constant fluctuation in density, and since the scale of the record is approximately logarithmic in density the change of deflection for a given small increase becomes less as the mean density increases.

For this reason the full sensitivity of the instrument cannot always be used. The limiting value can readily be found in practice. It is such that any further increase in sensitivity would make the drawing of a smooth line through the " clear-plate " curve impossible or highly arbitrary. In practice it is found that the amplitude of the variations may be as much as five scale divisions, and a good value of the background line be obtained.

It must be remembered that this variation is only about 0·01 in density, so that the measurement of densities in the exposed part of the plate where the density is 1·0 to 2·0, say, will be little in error if this fluctuation can be smoothed out and a line drawn to an error of 0·001. It is, however, important to notice that if the photometer is being used to detect faint companions of lines or suspected close multiplets or to resolve Zeeman patterns, it may be for the purpose unimportant to observe the background intensity, in which case greater freedom can be allowed, but care must be taken that the sensitivity is not made so great that the random fluctuations are of the same order as the pattern to be measured. In other words, magnification in a direction perpendicular to the direction of recording is not the paramount thing; a smooth record of smaller amplitude may give more information.

Stellar Parallaxes Determined at the Royal Observatory, Greenwich.

(*Communicated by the Astronomer Royal.*)

The following list is a continuation of the results obtained with the 26-inch telescope, and published annually in the November number of the *Monthly Notices*.

In the column " Proper Motion " the first value given is from meridian observations; the second is the relative proper motion derived from the photographs. The motions in both R.A. and Dec. are expressed in arc of a great circle.

In calculating the parallax the relative proper motion has been used, *not* the meridian proper motion.

In computing the P.E. an average value $\pm''\cdot025$ has been adopted as the probable error of a single plate of unit weight. This value has been derived from a discussion of the residuals of 333 determinations

of parallax, each of which depends on 23 plates on the average. The P.E. has accordingly been calculated by the formula

$$r = \frac{\pm ''\cdot 025}{\sqrt{w}}.$$

The first P.E. given is that of the parallax, the second is the P.E. of the relative proper motion.

Generally six comparison stars of mean magnitude 11-12 have been used, of which good images are obtained with an exposure of three minutes. The magnitude of the parallax star has been reduced to equality by means of the rotating sector and (for the brighter stars) local desensitisation by means of a weak solution of copper sulphate.

The Visual Magnitude is taken from the *Bonn Durchmusterung*, the Photographic from the *Greenwich Astrographic Catalogue*.

No.	Name and B.D. No.	Mag. Vis.	Mag. Photog.	Spectral Type.	R.A. Dec. 1900·0.	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
					h m s	″	″	″	″	
1	O.A. 22		7·0	Ko	0 3 59	+·132	+·040	+·033	±·007	26
	64° 3		7·9		+64° 31′	+·113			±·006	
2	71° 31		9·3		0 37 14	+·328	+·107	+·001	±·007	26
			10·2		+71° 38′	+·330			±·006	
3	C. 136		8·6	G	0 54 42	−·192	−·053	+·037	±·006	27
	81° 27		8·7		+81° 34′	−·204			±·006	
4	SU Cas.		6·5	F5	2 43 3	+·015	+·002	−·011	±·007	22
	68° 200		Var.		+68° 28′	+·001			±·007	
5	73° 207		9·5	Ko	3 48 27	+·030	−·136	+·023	±·006	33
			10·1		+73° 39′	+·050			±·003	
6	43 Cam.		5·2	B5	6 42 55	+·003	+·003	−·003	±·006	26
	69° 394		5·1		+69° 0′	+·009			±·006	
7	O.A. 12060		8·6	F8	11 41 37	+·189	−·078	+·003	±·007	25
	77° 440		9·0		+77° 36′	+·176			±·007	
8	L.F. 2048		7·7	G5	12 13 19	−·120	+·022	+·003	±·008	21
	69° 657		8·1		+69° 21′	−·122			±·008	
9	7 Dra.		5·3	K5	12 43 29	+·002	−·002	+·028	±·008	23
	67° 764		6·7		+67° 20′	+·009			±·007	
10	C. 1945		8·6	G5	13 1 17	−·021	−·142	+·016	±·009	22
	85° 214		9·2		+84° 48′	+·018			±·008	
11	C. 2286		9·0	Ko	15 8 26	−·051	+·016	−·004	±·008	30
	83° 439		9·6		+83° 43′	−·006			±·005	

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type.	R.A. Dec. 1900·0.	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
				h m s	″	″	″	″	
12	γ UMi.	2·8	A2	15 20 53	−·012	+·012	+·006	±·007	31
	72° 679	3·3		+72° 11′	+·007			±·005	
13	Gr. 2347	5·5	K0	16 22 2	−·022	−·015	+·024	±·009	21
	69° 845	6·2		+69° 20′	−·026			±·008	
14	C. 2631	8·7	G5	17 27 11	+·008	+·003	+·005	±·009	24
	83° 505	8·9		+83° 25′	−·008			±·007	
15	C. 2693	8·9	G5	17 49 8	+·020	+·060	+·014	±·009	27
	84° 391	8·7		+84° 16′	−·004			±·007	
16	C. 2793	8·5	K0	17 51 34	+·020	+·025	+·045	±·008	28
	88° 105	8·8		+88° 44′	+·002			±·007	
17	40 Dra.	6·5	F5	18 7 32	+·053	+·122	+·046	±·010	25
	79° 570	6·6		+79° 59′	+·074			±·008	
18	41 Dra.	6·5	F5	18 7 38	+·045	+·114	+·031	±·010	25
	79° 571	6·4		+79° 59′	+·069			±·008	
19	80° 575	8·0	G5	18 22 19	+·021	+·042	+·024	±·009	29
		8·6		+80° 37′	+·056			±·006	
20	42 Dra.	5·0	K0	18 25 42	+·116	−·036	+·019	±·012	25
	65° 1271	6·0		+65° 30′	+·111			±·007	
21	C. 2871	9·4		18 44 27	+·021	+·183	+·035	±·009	27
	86° 279	10·2		+86° 47′	−·017			±·006	
22	L. 35695	5·5	K0	18 55 59	−·008	−·025	−·006	±·010	25
	65° 1309	6·4		+65° 7′	−·009			±·006	
23	Gr. 2917	6·7	F2	19 35 26	−·114	−·064	+·004	±·010	29
	71° 964	6·8		+71° 23′	−·116			±·004	
24	C. 2996	9·2		19 45 6	+·076	+·174	−·003	±·008	27
	81° 676	9·4		+81° 22′	+·085			±·008	
25	ε Dra.	3·8	K0	19 48 31	+·081	+·031	−·007	±·008	25
	69° 1070	4·6		+70° 1′	+·072			±·008	
26	64 Dra.	5·0	M*a*	20 0 25	−·001	−·016	+·007	±·008	27
	64° 1405	6·8		+64° 32′	+·002			±·006	
27	73 Dra.	5·5	A2*p*	20 32 50	+·008	−·012	·000	±·008	24
	74° 872	5·4		+74° 37′	+·029			±·007	
28	4 Cep.	5·5	A5	20 41 56	+·043	+·028	+·034	±·008	26
	66° 1318	5·9		+66° 18′	+·025			±·006	

No.	Name and B.D. No.	Mag. Vis. Photog.	Spectral Type.	R.A. Dec. 1900·0.	Proper Motion. R.A.	Proper Motion. Dec.	Parallax.	P.E.	No. of Plates.
				h m s	″	″	″	″	
29	α Cep.	2·6	A5	21 16 12	+·152	+·049	+·061	±·008	27
	61° 2111	2·7		+62° 10′	+·160			±·007	
30	11 Cep.	5·0	K0	21 40 27	+·114	+·101	−·008	±·007	25
	70° 1193	5·7		+70° 51′	+·114			±·008	
31	L.F. 4175	7·0	F2	22 23 43	+·069	+·070	+·029	±·009	21
	81° 775	7·0		+81° 26′	+·038			±·007	
32	65° 1846	9·4		23 0 31	+·308	−·109	+·031	±·008	28
		10·5		+66° 14′	+·307			±·004	
33	γ Cep.	3·5	K0	23 35 15	−·060	+·158	+·067	±·009	26
	76° 928	4·6		+77° 4′	−·049			±·005	

No. 25. Σ2603. A previous result, +″·030 ± ·009, for parallax was derived from plates taken through a neutral-tinted filter in conjunction with the rotating sector. The systematic character of the residuals (*Determinations of Stellar Parallax*, 1913–24, p. 113) shows that the previous determination is entitled to little weight. The same applies to the determination of the parallax of κ Cephei (p. 116). These are the only stars taken with this filter, and comparison of the residuals at corresponding times suggests that it has changed during its period of use.

Photometric Observations of Nova Persei, 1901. 1928–29; 1929–30.

By B. M. Peek.

Estimates of the magnitude of the above nova have been continued at Herne and Solihull during the last two years, the comparison stars used being the same as those cited formerly. The following table gives the Julian date and recorded magnitude:—

J.D.	Mag.	J.D.	Mag.	J.D.	Mag.
[2425]		486·67	12·22	551·30	12·96
439·60 (1928)	12·48	491·48	12·64±	554·40	13·45±
453·60	13·00	494·64	12·22	559·37	13·25
457·58	12·96	496·62	11·97	568·29	13·35
464·55	12·02	497·45	12·78 *	573·40	13·57
467·55	12·52	498·47	12·78	586·24	13·35
471·55	13·00	502·46	12·96	646·45 (1929)	13·77
472·62	12·12	504·43	13·09	650·42	13·62
473·62	12·17	506·55	13·25	655·36	13·37
477·60	12·32	545·28	13·08	656·40	13·25
485·67	12·22	548·49	13·02±	657·34	13·35

* It seems unlikely that the whole of this abrupt decrease in brightness was real. Such, however, would not be entirely without precedent, falls of $0^{m}\cdot8$ in 24 hours having been recorded by both Barnard and Steavenson.

J.D.	Mag.	J.D.	Mag.	J.D.	Mag.
671·31	13·67	893·42	13·32	975·42	12·53
675·31	13·60	897·39	13·37	982·46 (1930)	12·69
677·36	13·40±	899·40	13·35	983·38	12·54
678·34	13·30	901·40	13·35	985·47	12·45
680·35	13·38	911·37	12·74	992·40	12·53±
681·32	13·41	912·37	12·91	996·25	12·40
688·34	13·31	916·32	12·96	997·46	12·53
693·39	13·33	919·36	12·96	[2426]	
694·35	13·13	922·45	12·88	002·38	12·40
696·34	<13·2	924·50	12·64	017·35	12·9±
703·36	13·42	926·38	12·91	024·36	12·91
738·42	13·03	928·28	13·13	025·36	12·86
..	..	939·34	12·82	027·36	12·75
872·61	12·00	940·29	12·74±	028·36	12·74
876·44	13·38	947·40	12·48	037·39	12·82
879·40	13·33	948·25	12·64	039·41	12·82
880·40	13·23	950·24	12·64	052·35	13·06
881·43	13·11	951·29	12·54	060·38	12·98
882·42	13·05	965·23	12·78	065·40	12·91
883·43	13·28	967·27	12·69±		
891·41	13·37	972·24	12·0±		

On the Observation of Faint Meteors, as experienced in the case of those from the Orbit of Comet Schwassmann-Wachmann, 1930*d*. By Kaname Nakamura.

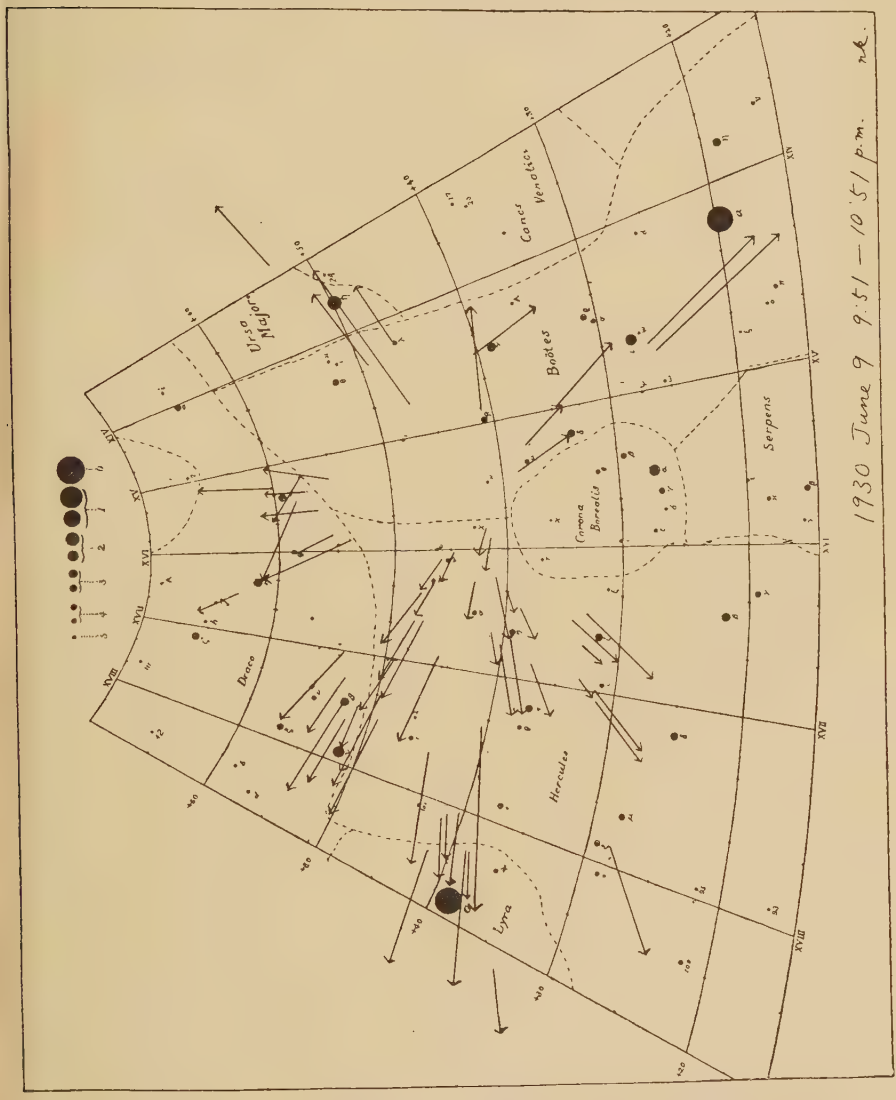
(*Communicated by Dr. A. C. D. Crommelin.*)

Upon receiving the Copenhagen telegram announcing the discovery of Comet Schwassmann-Wachmann in last May, micrometric observations of the daily positions were duly made at the Kwasan Observatory. Mr. Watanabe computed a set of parabolic elements of the preliminary orbit, and Mr. Shibata deduced therefrom the probable radiant point of the approaching meteoric shower, so that the whole programme of observations was established by May 20.* On May 21 a radiant point near ϵ Boötis was detected by observing scores of meteors in zenith. Computation of a probable parabolic orbit of these meteors, however, revealed that these were not what were expected in spite of some resemblances seen between the orbital elements of these meteors and the comet.†

Judging from the small inclination of the cometary orbit and from conditions similar to those of Pons-Winnecke's comet in past years, some early display of the meteors was strongly suggested. Just on May 24, when a careful watch for meteors in Boötes was being made, a radiant point was determined, being very fortunately suggested by a

* *Kwasan Bulletin*, **171**. † *Ibid.*, **172**, **173**.

stationary meteor at α 230°, δ + 48°. Although this was considerably preceding the comet's passage through its descending node, the possibility of this early meteoric display is supported by the results of later



calculation of the periodic orbit and also by previous experiences of the writer in the case of Pons-Winnecke's comet.

This meteoric shower was steadily followed toward the south-east for several subsequent nights. However, the work of observing the

comet itself prevented the writer from tracing the meteors very faithfully up to the end of May. The comet had attained its naked-eye visibility (perhaps as bright as $6^{m}\cdot5$) on the morning of May 29, and was actually seen with the naked eye the next morning. On May 31, when the sky was exceptionally clear, the comet appeared as a small, round nebulosity in the eastern part of Delphinus, the magnitude being slightly fainter than 5·5, fainter than, but similarly situated to, Winnecke's comet of 1927.

After the passing away of the comet the observations of meteors were resumed. But as number and brightness were not apparently increasing much, even as late as June 7, and the interference from the moonlight on the critical night was apparent, it was expected that there would not be a very rich display on June 9.* Notwithstanding, a watch was carefully kept, and the meteoric shower reached its climax of apparition in the evening of June 9. But most of those meteors were very faint, and a few of them were as bright as 4^{m}. Observations on later days were naturally not abundant because of the moonlight and also of the rainy season then beginning, but were nevertheless followed up to June 19. Yet it was fortunate indeed that the daily shifting of the radiant points was confirmed to a satisfactory degree. It is true that the present display was poor for the world-wide expectation suggested by the daily press. But if we think of so many known meteoric showers whose character of the moving radiants is really disappointing to amateurs, with practically a single exception of summer Perseids, it might be unnatural to expect real excitements in these meteoric displays, generally speaking. An apparition of only two or three meteors per hour is sometimes called a "shower." Many people frequently misunderstand the expression, thinking of true *showers* of fires and lights.

The following are the results of determined radiant points on each evening :—

Date.	Radiant Point.		No. of Meteors		Time of Duration.
	α	δ	all	bright (4).	
	h m	°			
1930 May 24	15 19	+48	..	..	..
25	22	48	..	..	..
June 3	29	46	..	..	..
6	35	43	..	..	..
7	38	42·5	..	..	..
9	45	41·5	59	12	60 m.†
	36	42			‡
10	40	42	36	3	30
12	48	41·5	..	..	..
13	52	40	..	..	..
19	16 18	39	..	..	..

* *Kwasan Bulletin,* **174**. † Near meteors. ‡ Distant meteors.

From these observed radiant points, seven preliminary parabolic elements were computed for the purpose of confirming whether they belong to the same group or not, as follows :—

Date.	Ω.	*i*.	ϖ.	Log *q*.
	°	°	°	
1930 May 25	64·1	28·1	200·6	9·992
June 3	72·3	26·8	200·4	9·992
9	78·0	26·1	201·3	9·992 *
	78·0	25·8	200·8	9·992 †
10	78·9	25·8	200·5	9·993
13	82·2	26·0	202·1	9·991
19	87·5	26·1	204·3	9·987

Although there are some systematic deviations between these elements and the cometary elements, general agreement is beyond doubt.

There will very often be similar cases in future, when the observation of faint meteors is keenly desired, and since so few people might have been successful in catching the present meteors, the writer will now try to describe his particular methods and experiences. This may be of some service to field observers in future. It seems to me that the observations of faint meteors will be extremely important for the development of meteoric astronomy.

Observations of meteors are usually made in very wide areas of the heavens, the radiant point of observed meteors being determined on the charts which contain relatively bright stars, possibly brighter than about the 5th magnitude. The number of meteors would be quite few per hour ordinarily, among which several meteors at most might belong to the same group, and, consequently, one or two radiants per night would be reliably determined. If we think of the wide area of the heavens which is *behind* the observer, the considerable number of those meteors which do not serve to determine the radiant points would be explained. Moreover, there may be some meteors which are erroneously included in a group for determining the radiant. Usually, at least four meteors are thought to be necessary to determine a radiant ; but really twice as many meteors are desirable for a reliable result.

Completeness in the study of meteoric swarms will be attained only by the co-operation of very many observers, so far as the present methods of observations are employed. Again, determinations of daily radiants would be the practical limit of attainment by field workers, so that the study of " moving radiants " is far from satisfactory. The cases of Perseids and meteors from Winnecke's comet (observations by Tashkent astronomers and my own series 1921–1930 agree in giving definite motion of the radiant from ζ Ursæ Majoris to β Boötis) are practically the only examples known as moving radiants.

Meteors fainter than 4th magnitude are very abundant. Observa-

* Near meteors. † Distant meteors.

tions on dates far from that of the climax display should be very carefully made in these faint meteors, for complete survey of the phenomena. Criticism is made that the inaccuracy of meteoric astronomy lies in the field observations themselves ; true, the observations with the small star-chart, containing stars as faint as 4th magnitude (although Young's recent charts containing $5^{m}\cdot0$ stars are improving the results), are making the results worse. If charts containing stars as faint as $5^{m}\cdot5$ are furnished and paths (even directions at least) of faint meteors are recorded accurately, the improvement in the results of meteoric observations would be considerable. These circumstances are especially needed when some limited fields of the heavens are watched for faint meteors naturally having short paths. Of course, detailed knowledge of the constellations by the observers is quite necessary.

According to the writer's long experience of those meteors from Winnecke's swarm, the faint meteors are unexpectedly numerous. Faint meteors are usually seen relatively near the radiant, probably because of their ignition at relatively high altitude and increased brightness caused by the foreshortening of paths, so that they are practically missed if we watch for bright meteors in the wider area. When a specially limited area of the heavens (some five degrees square) is watched with utmost caution, considerable numbers of faint meteors would be suspected. Most observers would reject these meteors as " difficult objects."

Observation would naturally yield their position and direction only. Assuming their reality, however, the radiant point would only be successfully determined if the closely neighbouring parts of the heavens were alternately watched. For exact determinations of the radiant, stationary or short and slow meteors are most effective. Meteors generally have their particular appearances according to their groups : for instance, the recent meteor paths from Comet Schwassmann-Wachmann were mostly short and thick, being somewhat spindle-shaped, so that they were easily distinguished from others. The azimuths of the streaks of meteors are also peculiar : the Schwassmann meteors made their appearance mostly toward β Draconis and Vega.

The writer's visual determinations of daily radiants of the Schwassmann-Wachmann meteors, without usual recordings upon star-charts, were similarly carried out. The uncertainties which are often ascribed to the faintness of meteors are effectively improved by observing excessive numbers of meteors. The writer is, in such cases, used to observe at least twenty meteors. Observations by recording are ordinarily limited to about 60 meteors per hour, beyond which it is practically impossible to go. In cases of faint and numerous meteors, the number observed per hour cannot be strictly relied upon. Accordingly, the date of maximum display can only be determined by observing mean magnitudes in case of continuous observations extending over several successive nights.

The observation of faint meteors usually causes considerable fatigue of eyes ; the writer, therefore, works mostly for 30 minutes,

or at most, 60 minutes. On a sheet of Young's charts, some sixty recordings make the sheet fairly full, and these are quite sufficient for determining the radiant.

Some experiments were made by the writer to test the efficiency of optical instruments, especially of very low-power telescope, in observing faint meteors. Galilean binoculars have too narrow fields for meteors, in spite of their unusual brightness. A special finder, with the magnification about $\times 3$ and equipped with a wide-angle eyepiece, will be good enough, if helped by excellent star-charts (such as Stucker's). On June 12 last the writer saw a fairly abundant display of meteors near the radiants, using his own 5-cm. Ottway finder (power $\times 8$). Telescopes do not add much to the brightness of these moving objects, so that extremely wide fields with low power are preferable.

The present description may seem somewhat unusual to some meteor observers, and be subject to their criticism. The writer's hope, however, is only to make some practical suggestions for future cases when faint meteors are expected.

Kwasan Observatory :
1930 *July.*

Note added to the above Paper.
By Issei Yamamoto, Director, Kwasan Observatory.

Mr. Nakamura was practically the only observer, among several other members of the staff of Kwasan who were watching the sky for the expected meteors in last May and June, who successfully observed the display. Quite recently, however, Mr. K. Siomi, a young member of the Society of Astronomical Friends, sent his reports to me, among which I found five meteors, seen on June 12–13, which are definitely radiating from a radiant point, R.A. 237°·5, Decl. + 41°, so that these undoubtedly belong to the shower from the comet 1930 *d*. Most of these are very faint. These circumstances make us think that Nakamura and Siomi are confirming each other's observations of these difficult objects. Mr. Siomi lives at Hukutiyama, a country town some 67 kilometres north-west of Kyoto. About a month ago Siomi visited Kwasan Observatory and spent a few weeks with us there. For some nights Siomi and Nakamura made simultaneous, and yet independent, observations of Lyrid meteors which had been detected by the latter some time before. The agreements of the results of both observers have made us extremely interested in the problem of these *faint meteors*, especially from the view-point of meteoric astronomy. Announcements of these observations are usually published immediately in the *Bulletins of the Kwasan Observatory* and also in *The Heavens* of the Society of A.F., while the detailed reports will be published in the *Publications of the Observatory*.

Kwasan Observatory, Kyoto :
1930 *September* 1.

Observations of the Comet Forbes (1930e), *made with the Doridis Refractor* (*Gautier* 40 *cm.*) *of the National Observatory of Athens.* By S. Plakidis.

Date. 1930.	G.M.T.	$\Delta\alpha$.	$\Delta\delta$.	Number of Comparisons.	$\alpha_{1930\cdot0}$.	log Parallax Factor.	$\delta_{1930\cdot0}$.	log Parallax Factor.
	h m s	m s	′ ″		h m s		° ′ ″	
June 23	22 6 30	+0 4·81	+4 6·0	12 : 10	19 32 30·87	9·314*n*	+ 0 9 44·8	0·730
24	20 57 14	−2 11·99	+4 47·7	10 : 10	19 19 27·55	9·475*n*	+ 2 23 43·6	0·713
25	20 33 0	−1 20·82	−3 8·5	10 : 10	19 6 30·88	9·491*n*	+ 4 35 39·3	0·695
26	21 29 56	−0 37·11	−6 32·5	10 : 10	18 53 23·26	9·261*n*	+ 6 46 36·0	0·662
27	21 3 48	−0 47·71	+1 23·8	10 : 10	18 41 35·04	9·308*n*	+ 8 41 30·8	0·641
28	21 19 13	+1 29·50	+1 32·8	10 : 10	18 30 2·41	9·148*n*	+10 30 16·2	0·611
July 1	21 44 16	+1 42·59	−5 11·7	10 : 10	17 59 30·36	7·561*n*	+15 0 41·1	0·532

Mean Positions of the Comparison Stars.

	Mag.	$\alpha_{1930\cdot0}$.	$\delta_{1930\cdot0}$	Catalogue.	
		h m s	° ′ ″		
1	7·2	19 32 26·06	+ 0 5 38·8	A.G. Nicolajew	4926
2	8·7	19 21 39·54	+ 2 18 55·9	,, Albany	6668
3	9·0	19 7 51·70	+ 4 38 47·8	,, ,,	6572
4	8·9	18 54 0·37	+ 6 53 8·5	,, Leipzig II.	8923
5	9·0	18 42 22·75	+ 8 40 7·0	,, Leipzig II.	8799
6	9·1	18 28 32·91	+10 28 43·4	B.D. +10°	3545
7	6·5	17 57 47·77	+15 5 52·8	A.G. Leipzig I.	6373

Remarks.—1930 June 23 : Diffuse nebulosity with a diameter of 67″. A nucleus of 14·5 mag. is visible from time to time.

June 24 : Circular nebulosity ; diameter of 80″. Nucleus of 14·0–14·5 mag. visible from time to time. The measurements were taken on the centre of the coma. The brightness of the comet is sensibly reduced when it passes near some stars more or less brilliant.

June 25 : The comet is hardly visible in the finder (80 mm.). Diameter of the coma about 120″. At times the coma seems a little elongated towards the S.S.E., with a nucleus of about 14·5 mag. excentric towards the N.N.W. Rapid motion.

June 26 : Same form. The comet is visible in the finder.

June 27 : Nucleus doubtful. Diameter of the coma about 100″.

June 28 : Nucleus doubtful ; there is rather a condensation, excentric towards the N.N.W., which gives the impression of a very faint tail, directed towards the S.S.E. Diameter of the coma about 115″.

July 1 : The comet is very faint. Nucleus doubtful. The coma is more extended, and about 120″ in diameter.

Athens :
1930 *October.*

Revision of the Measure of the Image of Pluto on the Uccle Plate of 1927 *January* 27.

Professor Stroobant writes to say that the remeasurement of the plate of 1927 January 27, on which the image of Pluto was found, was made by M. Delporte, and not by himself as stated in *M.N.*, **90**, 768.